CONVEX CONES OF n-MONOTONE FUNCTIONS

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PREFACE

The basic problem of this thesis is the determination of the extremal elements of the convex cone of n-monotone functions and the relationships that exist between these functions and the other elements of the cone. An n-monotone function is a real function f defined on [0,1] of the real line such that each of the first n differences of f is either nonnegative or nonpositive. The results of this study generalize some results of McLachlan [8] (numbers in square brackets refer to the bibliography at the end of the paper).

Chapter I gives the background associated with the problem and introduces the notation and terminology that is used throughout the study. In Chapter II the extremal elements of the convex cone A_n of functions alternating of order n are determined. These functions were defined by Choquet [4]. It is intended that the inclusion of Chapter II will provide insight into the more general discussion in Chapter III. In Chapter III the extremal elements of the cone of n-monotone functions are characterized. An integral representation of n-monotone functions in terms of the extremal elements is given in Chapter IV by using Choquet's Theorem (cf. [4], p. 237). Finally, Chapter V is a summary of the paper and lists some related problems which would be of interest for further consideration.

It was noted above that the A_n cones were considered in Chapter II in order to motivate the more general development. Another reason the

A_n cones are dealt with in detail is that they are closely related to the completely monotonic functions. In fact, a real function f defined on [0,1] is completely monotonic there if, and only if, $f(0) + f(1) - f$ is in A_∞, where A_∞ denotes the intersection of the A_n cones.

I wish to express my appreciation to all those who assisted me in pursuing my graduate studies and in the preparation of this thesis. In particular, I would like to thank Professor E. K. McLachlan for his invaluable guidance and encouragement. My thanks go to Professors J. Agnew, H. Uehara and D. Boyd for their encouragement and cooperation. Finally, my deepest thanks go to my wife, Carole, without whose encouragement and assistance I could never have completed my graduate studies.

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CHAPTER I

INTRODUCTION

This thesis considers certain classes of real functions defined on $[0,1]$ of the real line. These classes of functions, which are defined by finite differences, form convex cones in certain linear spaces of functions. Preliminary to the main discussion of the problem, some standard definitions and notation will be given.

Definition 1.1: Let A and B be subsets of a real linear space L, and let $\lambda \in R$. Then

$$A + B = \{x + y : x \in A \text{ and } y \in B\},$$

$$-A = \{x : -x \in A\},$$

$$A - B = A + (-B), \quad \text{and}$$

$$\lambda A = \{\lambda x : x \in A\}.$$

Definition 1.2: A set C in a real linear space L is a convex cone if 1) C is convex, 2) $\lambda C \subset C$ for all $\lambda \geq 0$ in R, and 3) $C \cap (-C) = \{\varphi\}$ where φ is the origin in L.

Note that condition 1 can be replaced by 1') $C + C \subset C$.

If K denotes the wedge shaped subset of E_2 as illustrated in Figure 1.1, then K is a convex cone. If $y \in K$ and y does not lie on the ray determined by x_1 nor the ray determined by x_2, then there are vectors y_1 and $y_2 \in K$ such that $y = y_1 + y_2$ and y_1 and y_2 are not scalar multiples of y. However, if $x_1 = y + z$, where y and $z \varepsilon K$, then y and z must be scalar multiples of x_1. Likewise, x_2 has this same property. This property of x_1 and x_2 is made precise in the following definition.

<u>Definition 1.3</u>: Let C be a convex cone in a real linear space L. An element $x \varepsilon C$ is called an <u>extremal element</u> of C if $x_1, x_2 \varepsilon C$ and $x_1 + x_2 = x$ imply that x_1 and x_2 are scalar multiples of x. An extremal element of C is said to be <u>extremal</u> in C.

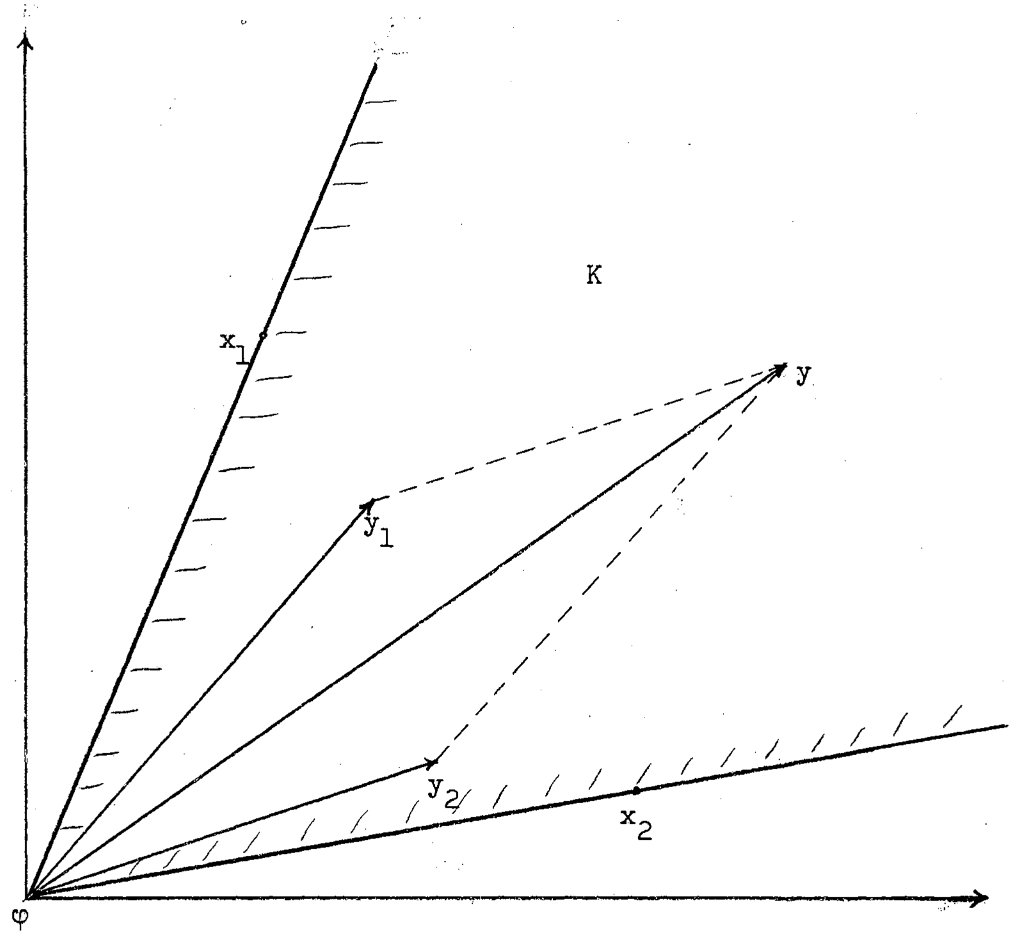

Figure 1.1.

Consider the cone C that is Figure 1.2(a). Let H be a hyperplane (that is, a translate of a two-dimensional subspace) which meets every ray of C but does not contain the origin. Then $S = H \cap C$ is a convex set (cf. Figure 1.2(b)).

Definition 1.4: Let S be a convex set in a real linear space L. An element $x \in S$ is called an extreme point of S if there do not exist two points y and z in S and a real number $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha)z$. The set of extreme points of S will be denoted by ext S.

Notice that x_i is both an extreme point of S and an extremal element of C, $i = 1,2,3$ (cf. Figure 1.2). If $x \in S$, then there are nonnegative scalars λ_1, λ_2, λ_3 such that

$$\sum_{i=1}^{3} \lambda_i = 1 \quad \text{and} \quad x = \sum_{i=1}^{3} \lambda_i x_i .$$

In other words, every point in S can be represented as a finite sum of extreme points of S. Since S meets every ray of C in exactly one point, every point in C is a unique scalar multiple of such a representation. Therefore, every point in C can be represented as a finite sum of extremal elements of C, since λx_i is an extremal element of C for $\lambda > 0$ and $i = 1,2,3$.

If C is a convex cone, then C-C is the smallest linear space containing C (cf. [3], p. 47). If the dimension of C-C is infinite, then the representation of points of C in terms of extremal elements of C is not quite as simple. In fact, the representation is no longer a finite

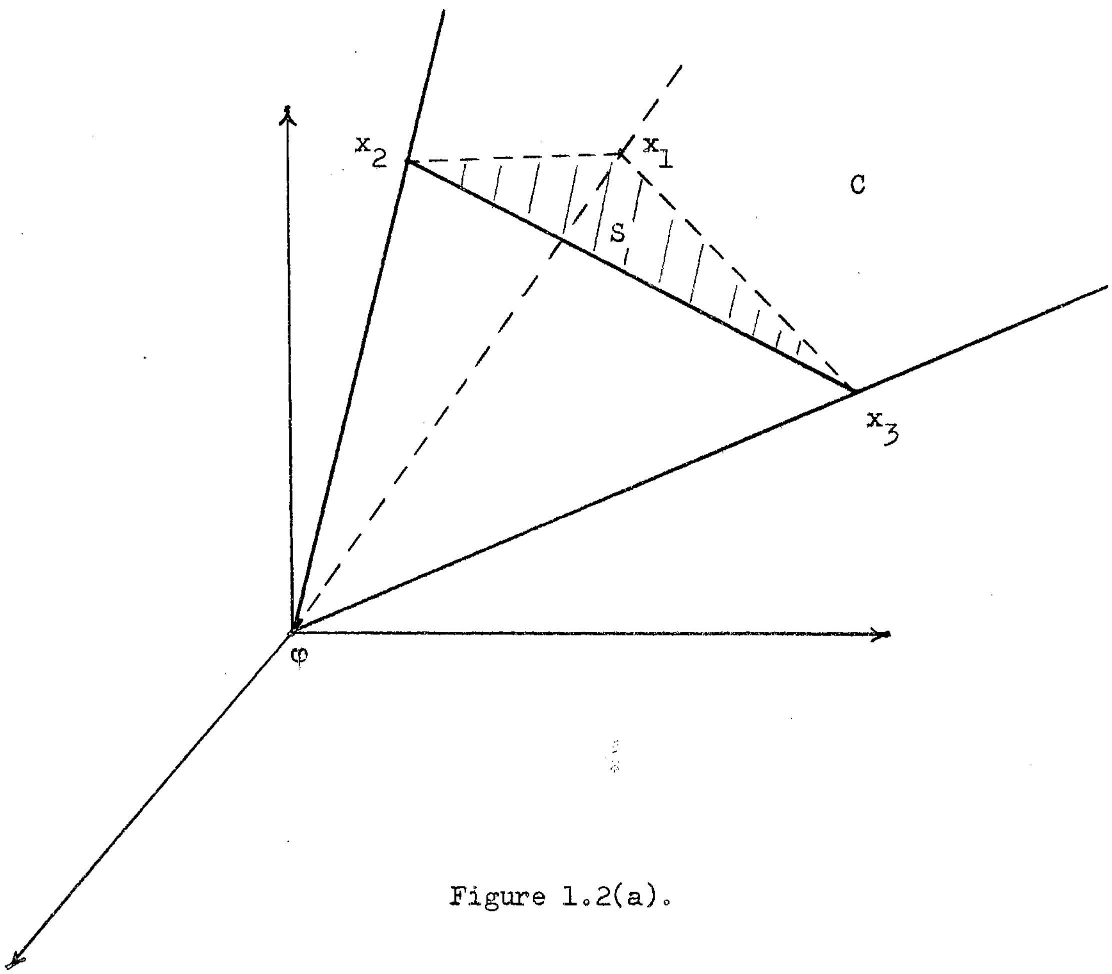

Figure 1.2(a).

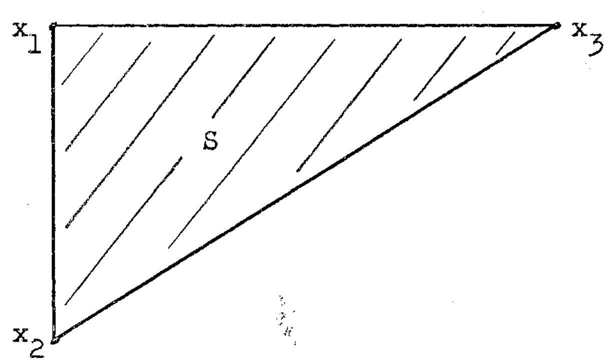

Figure 1.2(b).

sum but is instead an "infinite sum"; that is, an integral representation. This integral representation is made possible by Theorem 39.4 of Choquet [4]. An alternative form of this theorem, the form that is used in this study, is due to Phelps (cf. [9], p. 5). In infinite-dimensional spaces the extremal elements of a cone may be dense in the cone. In this case, the integral representation is of little value. Thus, in infinite-dimensional spaces the extremal elements may be "too numerous" to get a meaningful representation in terms of extremal elements.

If a given class C of functions forms a convex cone and the linear space C-C can be topologized in such a way that Choquet's Theorem will apply, then the functions in C are completely characterized, via the integral representation, once the extremal elements of C are determined. This thesis is concerned with real functions defined on the unit interval [0,1] which satisfy a certain set of difference inequalities. The interval [0,1] was chosen for convenience and could be replaced by any closed interval. The problem of finding the extremal elements of a convex cone of functions determined by a set of difference inequalities has been considered by Choquet (cf. [4], p. 249) and McLachlan [8]. In fact, it was McLachlan's paper [8] which provided the motivation for this study.

Before considering the problem of finding extremal elements, it is necessary to list some of the properties of the difference operator. These properties can be found in [2].

<u>Definition 1.5</u>: If f is a real-valued function defined on [0,1], then $\Delta_h^1 f(x) = f(x+h) - f(x)$ for $h > 0$ and $[x,x+h] \subset [0,1]$, and

$\Delta_h^n f(x) = \Delta_h^{n-1} f(x+h) - \Delta_h^{n-1} f(x)$ for $h > 0$ and $[x, x+nh] \subset [0,1]$, where $n > 1$.

It is easy to establish by induction the useful formula

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x+(k-j)h). \tag{1.1}$$

Then it easily follows that Δ operators are permutable; that is, $\Delta_{h_2}^{k_2}[\Delta_{h_1}^{k_1} f(x)] = \Delta_{h_1}^{k_1}[\Delta_{h_2}^{k_2} f(x)]$. The property of the operator Δ given in the following proposition is very useful in the proof of some of the results of both Chapters II and III.

<u>Proposition 1.1</u> (Lemma 1 in [2]): If f is a real-valued function defined on $[0,1]$ and

$$\Delta_h^k f(x) \geq 0,$$

where $k > 2$, $h > 0$ and $[x, x+kh] \subset [0,1]$, then for any k positive numbers $\delta_1, \delta_2, \ldots, \delta_k$

$$\Delta_{\delta_1}^1 \Delta_{\delta_2}^1 \cdots \Delta_{\delta_k}^1 f(x) \geq 0$$

provided that $0 \leq x < x+\delta_1+\delta_2+\cdots+\delta_k \leq 1$.

CHAPTER II

EXTREMAL ELEMENTS OF THE CONVEX CONE A_n OF FUNCTIONS

Let A_1 be the set of nonnegative real functions f on $[0,1]$ such that $\nabla_h^1 f(x) = f(x) - f(x+h) \leq 0$, $h > 0$, for $[x,x+h] \subset [0,1]$, and let A_n, $n > 1$ be the set of functions belonging to A_{n-1} such that $\nabla_h^n f(x) = \nabla_h^{n-1} f(x) - \nabla_h^{n-1} f(x+h) \leq 0$ for $[x,x+nh] \subset [0,1]$. Since $\nabla_h^k f(x) = (-1)^k \Delta_h^k f(x)$, $k \geq 1$, the analogue of Proposition 1.1 for the difference operator ∇ is valid. Since the sum of two functions in A_n belongs to A_n and since a nonnegative real multiple of an A_n function is an A_n function, the set of A_n functions forms a convex cone. It is the purpose of this chapter to give the extremal elements of this cone. Following the notation of Choquet, [4], a function in A_n is said to be alternating of order n on $[0,1]$. The intersection of the A_n cones, $\cap A_n$, is the class of functions which Choquet denoted as alternating of order ∞. Thus, the set of these functions, which will be denoted by A_∞, forms a convex cone also. The extremal elements for the convex cone A_∞ are given too.

Proposition 2.1: The extremal elements of A_1 are precisely the functions in A_1 which assume exactly one positive value in $[0,1]$.

Proof: For the function f such that $f(x) = 0$, $x \in [0,\xi)$, $f(x) = c > 0$, $x \in [\xi,1]$ where $0 \leq \xi \leq 1$ and $f = f_1 + f_2$ where

f_1 and $f_2 \in A_1$ then $0 = \nabla_h^1 f(x) = \nabla_h^1 f_1(x) + \nabla_h^1 f_2(x)$ implies $\nabla_h^1 f_i(x) = 0$ for $i = 1,2$ and $[x,x+h] \subset [\xi,1]$. Therefore, $f_i(x) = 0$, $x \in [0,\xi)$, $f_i(x) = c_i \geq 0$, $x \in [\xi,1]$, $i = 1,2$, where $c_1 + c_2 = c$. Hence, f is an extremal element of A_1.

If f assumes at least two positive values in $[0,1]$, then a nonproportional decomposition can be given by taking

$$f_1(x) = \min \{f(x), (1/2)[f(0)+f(1)]\}$$

and $f_2 = f - f_1$.

Extremal Elements of A_2

Since $\nabla_h^k f(x) = (-1)^k \Delta_h^k f(x)$ for $k \geq 1$, the functions of A_2 are exactly the nonnegative, nondecreasing and concave functions on $[0,1]$ (cf. [11], p. 148). Hence, if $f \in A_2$ then f'_- and f'_+ exist on $(0,1]$ and $(0,1)$, respectively. The left derivative f'_- is a nonnegative, nonincreasing, left-continuous function and f'_+ is nonnegative, nonincreasing and right-continuous (cf. [7], p. 4). In fact, $f'(x)$ exists for almost all $x \in (0,1)$ since $f'_-(x) \neq f'_+(x)$ if, and only if, f'_+ is discontinuous at x and f'_+ can have at most a countable number of discontinuities (cf. [5], p. 71).

Since a function in A_2 must be continuous on $(0,1]$, the only extremal elements of A_1 which are in A_2 are those functions f such that $f = c > 0$ on $(0,1]$ while $f(0) = 0$ or $f(0) = c$ and these functions are again extremal in A_2. If $f \in A_2$, f is not constant and $f(0) > 0$, then a nonproportional decomposition can be given by taking $f_1 = f(0)$ and $f_2 = f - f_1$. If $f \in A_2$, $f(0) = 0$, f is not constant on $(0,1]$ and f is not continuous at 0 (that is, $f(0+) > 0$), then take $f_1 = f(0+)$

on $(0,1]$, $f_1(0) = 0$ and $f_2 = f - f_1$. In so doing, f_1 and $f_2 \varepsilon A_2$ and f_1 and f_2 are not proportional to f. Since this same technique still can be used for A_n, $n > 2$, the only extremal elements of A_n such that $f(0) > 0$ are the positive constant functions, and the only extremal elements of A_n which are discontinuous at 0 are those functions f such that $f(0) = 0$ and $f = c > 0$ on $(0,1]$.

If $f \varepsilon A_2$ such that f'_- assumes exactly one positive value in $(0,1]$, then $f(x) = mx$, $x \varepsilon [0,\xi]$ and $m\xi$ for $x \varepsilon [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$. If f_1 and $f_2 \varepsilon A_2$ such that $f = f_1 + f_2$, then f_1 and f_2 are linear where f is linear, and f_1 and f_2 are constant where f is constant. Thus, f_1 and f_2 are proportional to f and f is therefore extremal. The next proposition shows that A_2 has no extremal elements other than those already mentioned.

Proposition 2.2: If $f \varepsilon A_2$ such that $f(0+) = f(0) = 0$ and f'_- assumes at least two positive values in $(0,1]$, then there exist two functions g and h in A_2 such that $f = g + h$ and g and h are not proportional to f.

Proof: Since f'_- is left-continuous at 1, there are numbers x_0 and x_1 such that $0 < x_0 < x_1 < 1$ and $f'_-(x_0) > f'_-(x_1) > 0$. Define

$$g(x) = f'_-(x_1)\ x,$$

$x \varepsilon [0,x_1]$ and

$$g(x) = f(x) - [f(x_1) - f'_-(x_1)x_1]$$

for $x \varepsilon [x_1,1]$ and let $h = f - g$. Since f'_- is nonnegative and g is continuous at x_1, then g is nonnegative and nondecreasing. Since

$$f'_-(x) \geq f'_-(x_1) = g'(x)$$

for $x \in (0, x_1]$, it follows that

$$h'(x) \geq 0$$

for $x \in (0,1]$ and h is nondecreasing. Thus, h is nonnegative, since $h(0) = 0$.

Since f is concave and $f(0) = 0$, it follows that

$$f(x) = \int_0^x f'_-(t)\, dt \tag{2.1}$$

for $x \in [0,1]$ (cf. [7], p. 5). If $x \in [0, x_1]$, then

$$g(x) = \int_0^x f'_-(x_1)\, dt = \int_0^x g'_-(t)\, dt.$$

If $x \in [x_1, 1]$, then

$$\int_0^x g'_-(t)\, dt = \int_0^{x_1} g'_-(t)\, dt + \int_{x_1}^x g'_-(t)\, dt = g(x_1) + \int_{x_1}^x f'_-(t)\, dt$$

and it follows from equation (2.1) that

$$\int_0^x g'_-(t)\, dt = f'_-(x_1)x_1 + f(x) - f(x_1) = g(x).$$

Thus,

$$g(x) = \int_0^x g'_-(t)\, dt \tag{2.2}$$

for $x \in [0,1]$. Since $h = f - g$, then

$$h(x) = f(x) - g(x) = \int_0^x [f'_-(t) - g'_-(t)]\, dt = \int_0^x h'_-(t)\, dt \tag{2.3}$$

for $x \in [0,1]$. Since f'_- is a nonincreasing function, then g'_- and h'_- are both nonincreasing. Therefore, it follows from equations (2.2) and (2.3) that g and h are concave on $[0,1]$ and g and h are in A_2 (cf. [7], p. 7).

By noting that $f(0) = 0$, f is strictly concave on $[0,x_1]$ and $f'_-(x_0) > f'_-(x_1)$, it can be shown that

$$\frac{g(x_1)}{f(x_1)} \neq \frac{g(x_0)}{f(x_0)},$$

and hence, g is not proportional to f.

Thus, the extremal elements of A_2 are the positive constant functions, the functions which are a positive constant on $(0,1]$ and zero at 0 and those f such that $f(x) = mx$, $x \in [0,\xi]$ and $m\xi$ for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$. Designate this latter function by $e(m,\xi,1;)$.

Extremal Elements of A_n, $n > 2$

It will be shown that the extremal elements of A_n, $n > 2$, are indefinite integrals of the extremal elements of a cone which is similar to A_2. This cone is given in the following definitions.

Definition 2.1: If g is a real continuous function on $(0,1]$ and n is a positive integer, then g is said to satisfy property $P(n)$ if

$$\lim_{\delta \to 0} \int_1^{\delta} \int_1^{t_{n-1}} \cdots \int_1^{t_2} \int_1^{t_1} g(t)\, dt\, dt_1 \ldots dt_{n-1}$$

exists and is finite.

Definition 2.2: Let $K_0(n)$ denote the convex cone of nonnegative real continuous functions g on $(0,1]$ such that $\nabla_h^k g(x) \geq 0$ for $[x, x+kh] \subset (0,1]$, $k = 1,2$, and g satisfies property $P(n)$.

Definition 2.3: Let $K_1(n)$ denote the convex cone of nonpositive real continuous functions g on $(0,1]$ such that $\nabla_h^k g(x) \leq 0$ for $[x, x+kh] \subset (0,1]$, $k = 1,2$, and g satisfies property $P(n)$.

The functions of $K_0(n)$ ($K_1(n)$) are exactly the nonnegative (nonpositive), nonincreasing (nondecreasing) and convex (concave) functions on $(0,1]$ which satisfy property $P(n)$. If $f(x) = 1-x$, $x \in [0,1]$, then $f \in K_0(n)$ and $-f \in K_1(n)$ and it follows that the cones $K_0(n)$ and $K_1(n)$ are both nonempty. The extremal elements of $K_0(n)$ and $K_1(n)$ are found in the following two lemmas.

Lemma 2.1: The extremal elements of $K_1(n)$ are the negative constant functions and the functions g such that $g(x) = m(x-\xi)$, $x \in (0,\xi]$ and $g(x) = 0$ for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$.

Proof: If $g = c < 0$ and $g = g_1 + g_2$, where g_1 and $g_2 \in K_1(n)$, then $0 = \nabla_h^1 g(x) = \nabla_h^1 g_1(x) + \nabla_h^1 g_2(x)$ implies $\nabla_h^1 g_i(x) = 0$ for $i = 1,2$ and $[x,x+h] \subset (0,1]$. Therefore, $g_i = c_i$, $c_i \leq 0$, $i = 1,2$, where $c_1 + c_2 = c$. Hence, g is an extremal element of $K_1(n)$. If $g \in K_1(n)$ such that g is not constant and $g(1) < 0$, then take $g_1 = g(1)$ and $g_2 = g - g_1$. In so doing, g_1 and $g_2 \in K_1(n)$, and g_1 and g_2 are not proportional to g. Thus, the only extremal elements of $K_1(n)$ which are negative at 1 are the negative constant functions.

If $g(x) = m(x-\xi)$, $x \in (0,\xi]$ and $g(x) = 0$ for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$, then, for $g = g_1 + g_2$, it follows that g_1 and g_2 are linear where g is linear, and g_1 and g_2 are zero where g is zero. Thus, g_1 and g_2 are proportional to g, and g is therefore extremal.

If $g \in K_1(n)$ such that $g(1) = 0$ and there are numbers x_0 and x_1 where $0 < x_0 < x_1 < 1$ and $g'_-(x_0) > g'_-(x_1) > 0$, then a nonproportional decomposition can be given by taking $g_1(x) = g'_-(x_1)(x-x_1) + g(x_1)$, $x \in (0,x_1]$, $g_1(x) = g(x)$ for $x \in [x_1,1]$ and $g_2 = g - g_1$. The proof that g_1 and g_2 are nonpositive, nondecreasing, concave and not proportional to g is essentially the same as the proof of Proposition 2.2 and is not given here. The only fact that remains to be shown is that g_1 and g_2 satisfy property P(n). Since $g_1(0+) = g(x_1) - g'_-(x_1)x_1$ and g_1 is continuous on $(0,1]$, it follows that g_1 satisfies property P(n). Since $g \in K_1(n)$ and $g_2 = g - g_1$, then g_2 also satisfies property P(n). Hence, g_1 and $g_2 \in K_1(n)$.

Lemma 2.2: The extremal elements of $K_0(n)$ are the positive constant functions and the functions g such that $g(x) = m(\xi-x)$, $x \in (0,\xi]$ and $g(x) = 0$ for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$.

Proof: The lemma follows from Lemma 2.1 and the fact that

$K_0(n) = -K_1(n)$.

It is shown in the next two lemmas how the A_n functions are related to the functions in $K_0(n-2)$ and $K_1(n-2)$, where $n > 2$. These results will be used in finding the extremal elements of A_n.

Lemma 2.3: If $f \in A_n$, then $f^{(n-2)} \in K_0(n-2)$ if n is odd, and $f^{(n-2)} \in K_1(n-2)$ if n is even, where $n > 2$.

Proof: The proof will be by induction on n. If $f \in A_3$, then $f \in A_2$ and $\nabla_h^3 f(x) \leq 0$ for $[x,x+3h] \subset [0,1]$, and it follows from Proposition 1.1 that

$$f(x) - [2f(x+h) + f(x+\delta)] + [2f(x+h+\delta) + f(x+2h)] - f(x+2h+\delta)$$

$$= \nabla_h^2 \nabla_\delta^1 f(x) \leq 0,$$

where $h > 0$, $\delta > 0$ and $[x,x+2h+\delta] \subset [0,1]$ (cf. equation 1.1). Hence,

$$-\frac{f(x+\delta) - f(x)}{\delta} + 2\,\frac{f(x+h+\delta) - f(x+h)}{\delta} - \frac{f(x+2h+\delta) - f(x+2h)}{\delta} \leq 0,$$

which implies that

$$f_+'(x) - 2f_+'(x+h) + f_+'(x+2h) \geq 0.$$

Therefore, f_+' is convex on $(0,1)$, which implies that f_+' is continuous there, and it follows that $f' = f_+'$ on $(0,1)$. Since

$$f_-'(1) = f_-'(1-) = \lim_{x \to 1-} f_-'(x) = \lim_{x \to 1-} f'(x),$$

f' is continuous on $(0,1]$ by defining $f'(1) = f'(1-)$, and f' is nonnegative, nonincreasing and convex. Also

$$\int_0^1 f'(t)\,dt = f(1) - f(0);$$

that is, f' satisfies property $P(1)$ and thus, $f' \in K_0(1)$.

Assume that $f \in A_n$ implies $f^{(n-2)} \in K_0(n-2)$ if n is odd and $f^{(n-2)} \in K_1(n-2)$ if n is even, where $n > 2$. If $f \in A_{n+1}$, then

$$\nabla_h^3 \nabla_h^{n-2} f(x) = \nabla_h^{n+1} f(x) \leq 0$$

for $[x, x+(n+1)h] \subset [0,1]$, which implies that

$$\nabla_h^3 \nabla_{\delta_1}^1 \nabla_{\delta_2}^1 \cdots \nabla_{\delta_{n-2}}^1 f(x) \leq 0$$

for $[x, x+3h+\delta_1+\delta_2+\cdots+\delta_{n-2}] \subset [0,1]$ (cf. Proposition 1.1). It then follows that

$$(-1)^{n-2} \nabla_h^3 f^{(n-2)}(x) \leq 0$$

for $[x, x+3h] \subset (0,1]$. Therefore,

$$(-1)^{n-2} \nabla_h^2 \nabla_\delta^1 f^{(n-2)}(x) \leq 0 \tag{2.4}$$

for $h > 0$, $\delta > 0$ and $[x, x+2h+\delta] \subset (0,1]$.

If n is odd, then $f^{(n-2)}$ is nonnegative, nonincreasing and convex, and it can be shown that $f_-^{(n-1)}$ and $f_+^{(n-1)}$ exist on $(0,1]$ and $(0,1)$, respectively; these derivatives are nonpositive and nondecreasing and $f_+^{(n-1)}(x) \neq f_-^{(n-1)}(x)$ if, and only if, $f_+^{(n-1)}$ is discontinuous at x. Since inequality (2.4) implies

$$f^{(n-2)}(x) - [2f^{(n-2)}(x+h) + f^{(n-2)}(x+\delta)]$$
$$+ [2f^{(n-2)}(x+h+\delta) + f^{(n-2)}(x+2h)] - f^{(n-2)}(x+2h+\delta) \geq 0$$

for $h > 0$, $\delta > 0$ and $[x,x+2h+\delta] \subset (0,1]$, then

$$-\frac{f^{(n-2)}(x+\delta) - f^{(n-2)}(x)}{\delta} + 2\,\frac{f^{(n-2)}(x+h+\delta) - f^{(n-2)}(x+h)}{\delta}$$
$$-\frac{f^{(n-2)}(x+2h+\delta) - f^{(n-2)}(x+2h)}{\delta} \geq 0,$$

which implies that

$$f_+^{(n-1)}(x) - 2f_+^{(n-1)}(x+h) + f_+^{(n-1)}(x+2h) \leq 0.$$

Therefore, $f_+^{(n-1)}$ is concave, which implies that $f_+^{(n-1)}$ is continuous on $(0,1)$, and it follows that $f^{(n-1)} = f_+^{(n-1)}$ on $(0,1)$. Since

$$f_-^{(n-1)}(1) = f_-^{(n-1)}(1-) = \lim_{x\to 1-} f_-^{(n-1)}(x) = \lim_{x\to 1-} f^{(n-1)}(x),$$

$f^{(n-1)}$ is continuous on $(0,1]$ by defining $f^{(n-1)}(1) = f^{(n-1)}(1-)$, and $f^{(n-1)}$ is nonpositive, nondecreasing and concave.

If n is even, then $-f^{(n-2)}$ is nonnegative, nonincreasing and convex, and by inequality (2.4),

$$\nabla_h^2 \nabla_\delta^1 [-f^{(n-2)}(x)] = -\nabla_h^2 \nabla_\delta^1 f^{(n-2)}(x) \geq 0$$

for $h > 0$, $\delta > 0$ and $[x,x+2h+\delta] \subset (0,1]$. It follows from the argument given above that $-f^{(n-1)}$ is nonpositive, nondecreasing and concave; that is, $f^{(n-1)}$ is nonnegative, nonincreasing and convex.

It remains only to show that $f^{(n-1)}$ satisfies property P(n-1).

If $f \in A_{n+1}$, then $f \in A_n$, and

$$\begin{aligned} &\lim_{\delta \to 0} \int_1^\delta \int_1^{t_{n-2}} \cdots \int_1^{t_2} \int_1^{t_1} f^{(n-1)}(t)\, dt\, dt_1 \cdots dt_{n-2} \\ &= \lim_{\delta \to 0} \int_1^\delta \int_1^{t_{n-2}} \cdots \int_1^{t_3} \int_1^{t_2} f^{(n-2)}(t_1)\, dt_1\, dt_2 \cdots dt_{n-2} \\ &\qquad - f^{(n-2)}(1) \int_1^0 \int_1^{t_{n-2}} \cdots \int_1^{t_3} \int_1^{t_2} dt_1\, dt_2 \cdots dt_{n-2} \end{aligned}$$

exists and is finite, since $f^{(n-2)}$ satisfies property P(n-2) by the induction hypothesis.

The following definition is given to simplify the notation in the proofs of Lemma 2.4 and subsequent lemmas.

Definition 2.4: If g is a real continuous function on $(0,1]$ which satisfies property P(n), then define the function $I(g,n;)$ by the equation

$$I(g,1;x) = \int_0^x g(t)\, dt,$$

$$I(g,n;x) = \int_0^x \int_1^{t_{n-1}} \cdots \int_1^{t_2} \int_1^{t_1} g(t)\, dt\, dt_1 \cdots dt_{n-1},$$

$n = 2,3,4,\ldots,$ for $x \in [0,1]$.

Lemma 2.4: For $k \geq 1$, if $g \in K_0(2k-1)$ then $I(g,2k-1;) \in A_{2k+1}$, and if $g \in K_1(2k)$ then $I(g,2k;) \in A_{2k+2}$.

Proof: Since $K_1(2k) = -K_0(2k)$, it is sufficient to prove that $I(g,n-2;) \varepsilon A_n$, $n > 2$, if $(-1)^{n-1}g \varepsilon K_0(n-2)$. This proof will be by induction on n. If $g \varepsilon K_0(1)$, then $I(g,1;)$ is nonnegative on [0,1] since

$$I(g,1;x) = \int_0^x g(t)\, dt \geq 0,$$

for $x \varepsilon [0,1]$. If $[x,x+h] \subset [0,1]$, then

$$\nabla_h^1 I(g,1;x) = \int_{x+h}^x g(t)\, dt = (-h)g(\xi) \leq 0,$$

since $g(\xi) \geq 0$, where $x < \xi < x+h$. Thus, for $h > 0$ and $[x,x+kh] \subset [0,1]$, where $k = 2,3$,

$$\nabla_h^k I(g,1;x) = \nabla_h^{k-1} \nabla_h^1 I(g,1;x) = (-h)\nabla_h^{k-1} g(\xi) \leq 0,$$

since $\nabla_h^{k-1} g(\xi) \geq 0$ for $k = 2,3$. Hence, $I(g,1;) \varepsilon A_3$ whenever $g \varepsilon K_0(1)$.

Assume that $I(g,n-2;) \varepsilon A_n$ for $(-1)^{n-1}g \varepsilon K_0(n-2)$ and $n > 2$. If $(-1)^n g \varepsilon K_0(n-1)$, then let

$$f(x) = \int_1^x g(t)\, dt,$$

for $x \varepsilon (0,1]$. Since $(-1)^n g \varepsilon K_0(n-1)$, it follows that

$$(-1)^{n-1}f(x) = \int_1^x (-1)^{n-1}g(t)\,dt = -\int_1^x (-1)^n g(t)\,dt \geq 0$$

for $x \in (0,1]$,

$$\nabla_h^1 (-1)^{n-1}f(x) = (-1)^{n-1}\int_{x+h}^x g(t)\,dt = h(-1)^n g(\xi) \geq 0$$

for $0 < x < \xi < x+h \leq 1$ and

$$\nabla_h^2 (-1)^{n-1}f(x) = \nabla_h^1 \nabla_h^1 (-1)^{n-1}f(x) = h\nabla_h^1 (-1)^n g(\xi) \geq 0$$

for $0 < x < \xi < x+2h \leq 1$. The function f also satisfies property P(n-2) since

$$\lim_{\delta \to 0} \int_1^\delta \int_1^{t_{n-3}} \cdots \int_1^{t_2} \int_1^{t_1} f(t)\,dt\,dt_1 \cdots dt_{n-3}$$

$$= \lim_{\delta \to 0} \int_1^\delta \int_1^{t_{n-3}} \cdots \int_1^{t_1} \int_1^{t_0} g(t)\,dt\,dt_0 \cdots dt_{n-3}$$

and g satisfies property P(n-1). Hence, $(-1)^{n-1}f \in K_0(n-2)$ and it follows from the induction hypothesis that

$$I(g,n-1;) = I(f,n-2;) \in A_n.$$

By a repeated application of the mean value theorem for a Riemann integral, it can be shown that

$$\nabla_h^{n-1} I(g,n-1;x) = (-h)^{n-1}g(\xi)$$

for $[x,x+(n-1)h] \subset [0,1]$, where $x < \xi < x+(n-1)h$. It follows that

$$\nabla_h^{n+1} I(g,n-1;x) = \nabla_h^2 (-h)^{n-1}g(\xi) = -h^{n-1} \nabla_h^2 (-1)^n g(\xi) \leq 0$$

for $[x,x+(n+1)h] \subset [0,1]$, since $\nabla_h^2 (-1)^n g(\xi) \geq 0$. This inequality, together with the fact that $I(g,n-1;) \in A_n$, implies that $I(g,n-1;) \in A_{n+1}$.

In the following proposition, extremal elements of A_n are found by integrating the extremal elements of either $K_0(n-2)$ or $K_1(n-2)$.

Proposition 2.3: The function f such that $f(x) = m[\xi^{n-1}-(\xi-x)^{n-1}]$ for $x \in [0,\xi]$ and $m\xi^{n-1}$ for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$, is an extremal element of A_n, $n > 2$.

Proof: If f is such a function then

$$f^{(n-2)}(x) = (-1)^{n-1}m(n-1)!(\xi-x),$$

$x \in (0,\xi]$ and 0 for $x \in [\xi,1]$, and it follows from Lemmas 2.1 and 2.2 that $f^{(n-2)}$ is an extremal element of $K_0(n-2)$ if n is odd, whereas $f^{(n-2)}$ is extremal in $K_1(n-2)$ if n is even. Since $f(0) = 0$ and $f^{(k)}(1) = 0$ for $1 \leq k \leq n-3$, then $f = I(f^{(n-2)},n-2;)$, and it follows from Lemma 2.4 that $f \in A_n$.

If n is an odd integer and f_1 and $f_2 \in A_n$ such that $f = f_1 + f_2$, then $f_1^{(n-2)}$ and $f_2^{(n-2)} \in K_0(n-2)$ and $f^{(n-2)} = f_1^{(n-2)} + f_2^{(n-2)}$. Since $f^{(n-2)}$ is extremal in $K_0(n-2)$, there are constants $\lambda_i \geq 0$, $i = 1,2$, such that $f_i^{(n-2)} = \lambda_i f^{(n-2)}$. Since $f(0) = 0$ and $f^{(k)}(1) = 0$ for $1 \leq k \leq n-3$, it follows that $f_i(0) = f_i^{(k)}(1) = 0$ for $i = 1,2$ and $1 \leq k \leq n-3$. Hence,

$$f_i = I(f_i^{(n-2)},n-2;) = I(\lambda_i f^{(n-2)},n-2;) = \lambda_i I(f^{(n-2)},n-2;) = \lambda_i f,$$

$i = 1,2$, and f is therefore extremal.

On the other hand, if n is even then, as in the first part of the proof, f is extremal in A_n since $f^{(n-2)}$ is an extremal element of $K_1(n-2)$. Thus, if $f(x) = m[\xi^{n-1}-(\xi-x)^{n-1}]$, $x \varepsilon [0,\xi]$ and $m\xi^{n-1}$ for $x \varepsilon [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$, then f is extremal in A_n, $n > 2$. Denote this latter function by $e(m,\xi,n-1;)$.

The remaining extremal elements of A_n, $n > 2$, are given in the next proposition.

<u>Proposition 2.4</u>: If $m > 0$, the function $e(m,1,k;)$ is an extremal element of A_n for $n > 2$ and $1 \leq k < n-1$.

Proof: Since A_n is a subcone of A_{k+1} and $e(m,1,k;)$ is an extremal element of A_{k+1}, it is sufficient to show that $e(m,1,k;) \varepsilon A_n$. If $f = e(m,1,k;)$, then $f = I(f^{(k-1)},k-1;)$, where

$$f^{(k-1)}(x) = (-1)^k m(k!)(1-x)$$

for $0 < x \leq 1$ (cf. proof of Proposition 2.3). It follows from a repeated application of the mean value theorem for a Riemann integral that $\nabla_h^{k-1} f(x) = (-h)^{k-1} f^{(k-1)}(\xi)$, for $h > 0$, $[x,x+(k-1)h] \subset [0,1]$, where $x < \xi < x+(k-1)h$. Since $f^{(k-1)}$ is linear,

$$\nabla_h^{k+1} f(x) = \nabla_h^2 \nabla_h^{k-1} f(x) = (-h)^{k-1} \nabla_h^2 f^{(k-1)}(\xi) = 0 \qquad (2.5)$$

for $h > 0$, $[x,x+(k+1)h] \subset [0,1]$, and thus, $\nabla_h^p f(x) = 0$ for $h > 0$, $[x,x+ph] \subset [0,1]$ and $p \geq k+1$. Hence, $f \varepsilon A_n$ for every n, which implies that f is extremal in A_p, for $p \geq k+1$.

To this point it has been shown that the positive constant functions, the functions which are constant on $(0,1]$ and zero at 0, the functions $e(m,1,k;)$, where $m > 0$ and $1 \leq k \leq n-2$, and the functions $e(m,\xi,n-1;)$, where $m > 0$ and $0 < \xi \leq 1$, are extremal elements of A_n, for $n > 2$. The following three lemmas will prove that no other functions in A_n are extremal elements.

Lemma 2.5: Let $f \in A_n$, $n > 2$, such that $f(0+) = f(0) = 0$ and $f \neq e(m,1,k;)$, where $m > 0$ and $1 \leq k \leq n-3$. If there is an integer k such that $1 \leq k \leq n-3$ and $f^{(k)}(1) \neq 0$, then f is not an extremal element of A_n.

Proof: Let k denote the smallest integer such that $f^{(k)}(1) \neq 0$. Then $f \in A_n \subset A_{k+3}$ implies that $f^{(k+1)} \in K_0(k+1)$ if k is even, whereas $f^{(k+1)} \in K_1(k+1)$ if k is odd, and it follows from Lemma 2.4 that $I(f^{(k+1)},k+1;) \in A_{k+3}$. Since $f(0) = 0$ and $f^{(p)}(1) = 0$ for $1 \leq p < k$, then

$$I(f^{(k+1)},k+1;) = I(f^{(k)},k;) - f^{(k)}(1)I(1,k;) = f - e(m,1,k;),$$

where $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1) > 0$, because $(-1)^{k-1}f^{(k)} \in K_0(k)$ and $f^{(k)}(1) \neq 0$ imply $(-1)^{k-1}f^{(k)}(1) > 0$. Since $\nabla_h^p\, e(m,1,k;x) = 0$ for $h > 0$, $[x,x+ph] \subset [0,1]$ and $p \geq k+1$ and $f \in A_n$, it follows that

$$\nabla_h^p\, I(f^{(k+1)},k+1;x) = \nabla_h^p\, f(x) \leq 0,$$

for $[x,x+ph] \subset [0,1]$, $k+1 \leq p \leq n$ (cf. equation 2.5). Hence, $f - e(m,1,k;) \in A_n$, where $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1)$, and a nonproportional decomposition of f can be given by taking $f_1 = e(m,1,k;)$ and $f_2 = f - f_1$. Thus, f is not extremal.

Lemma 2.6: Let $f \varepsilon A_n$, $n > 2$, such that $f \neq 0$, $f(0+) = f(0) = 0$ and $f \neq e(m,1,k;)$, where $m > 0$ and $1 \leq k \leq n-3$. If $f^{(n-2)} = 0$ on $(0,1]$, then f is not an extremal element of A_n.

Proof: If $f^{(n-2)} = 0$, then there is a positive integer $k \leq n-3$ such that $f^{(k)} \neq 0$ and $f^{(k)}$ is constant on $(0,1]$. Thus, $f^{(k)}(1) \neq 0$ and it follows from Lemma 2.5 that f is not extremal.

It is a consequence of Lemmas 2.5 and 2.6 that if f is an extremal element of A_n, $n > 2$, such that $f(0+) = f(0) = 0$ and either $f^{(n-2)} = 0$ or $f^{(k)}(1) \neq 0$ for some k, $1 \leq k \leq n-3$, then $f = e(m,1,k;)$, where $m > 0$ and $1 \leq k \leq n-3$.

Lemma 2.7: Let $f \varepsilon A_n$, $n > 2$, such that $f(0+) = f(0) = 0$, $f^{(n-2)} \neq 0$ and $f^{(k)}(1) = 0$ for $1 \leq k \leq n-3$. If f is an extremal element of A_n, then $f = e(m,1,n-2;)$ or $f = e(m,\xi,n-1;)$, where $m > 0$ and $0 < \xi \leq 1$.

Proof: Since $f(0) = 0$ and $f^{(k)}(1) = 0$ for $1 \leq k \leq n-3$, then $f = I(f^{(n-2)},n-2;)$. If n is odd then $f^{(n-2)} \varepsilon K_0(n-2)$. If g_1 and $g_2 \varepsilon K_0(n-2)$ such that $f^{(n-2)} = g_1 + g_2$, then

$$f = I(f^{(n-2)},n-2;) = I(g_1+g_2,n-2;) = I(g_1,n-2;) + I(g_2,n-2;).$$

Then $f_i = I(g_i,n-2;)$, $i = 1,2$, implies that f_1 and $f_2 \varepsilon A_n$ and $f = f_1 + f_2$. Since f is extremal in A_n, there are numbers $\lambda_i \geq 0$ such that $f_i = \lambda_i f$, $i = 1,2$, which implies that $g_i = f_i^{(n-2)} = \lambda_i f^{(n-2)}$,

$i = 1, 2$, and $f^{(n-2)}$ is therefore extremal in $K_0(n-2)$. If $f^{(n-2)} = c > 0$, then $f = I(c, n-2;) = e(m, 1, n-2;)$, where $m = (-1)^{n-1}c/(n-2)! > 0$. If $f^{(n-2)}(x) = c(\xi - x)$, $x \varepsilon (0, \xi]$ and 0 for $x \varepsilon [\xi, 1]$, where $c > 0$ and $0 < \xi \leq 1$, then $f = I(f^{(n-2)}, n-2;) = e(m, \xi, n-1;)$, where $m = (-1)^{n-1}c/(n-1)! > 0$.

On the other hand, if n is even, then $f^{(n-2)} \varepsilon K_1(n-2)$, and an argument similar to that above shows that $f^{(n-2)}$ is an extremal element of $K_1(n-2)$. If $f^{(n-2)} = c < 0$, then $f = I(c, n-2;) = e(m, 1, n-2;)$, where $m = (-1)^{n-1}c/(n-2)! > 0$. If $f^{(n-2)}(x) = c(\xi - x)$, $x \varepsilon (0, \xi]$ and 0 for $x \varepsilon [\xi, 1]$, where $c < 0$ and $0 < \xi \leq 1$, then $f = I(f^{(n-2)}, n-2;)$ $= e(m, \xi, n-1;)$, where $m = (-1)^{n-1}c/(n-1)! > 0$.

Therefore, the extremal elements of A_n, $n > 2$, are the positive constant functions, the functions which are a positive constant on $(0, 1]$ and zero at 0, the functions $e(m, 1, k;)$, where $m > 0$, $1 \leq k \leq n-2$, and the functions $e(m, \xi, n-1;)$, where $m > 0$ and $0 < \xi \leq 1$.

Extremal Elements of A_∞

It has already been noted that $e(m, 1, k;)$ is an extremal element of A_n for each $n > k$ (cf. Proposition 2.4). It follows that $e(m, 1, k;)$ is an extremal element of A_∞ for every positive integer k. It is shown in the following proposition that A_∞ has no other extremal elements which are continuous and zero at 0.

<u>Proposition 2.5</u>: If $f \varepsilon A_\infty$ such that $f(0+) = f(0) = 0$ and $f \neq e(m, 1, k;)$, where $m > 0$ and k is a positive integer, then f is not an extremal element of A_∞.

Proof: Since $f \in A_\infty$ is a function of class C^∞ on $(0,1]$, it follows from a theorem of Bernstein, Theorem 13-31 in [1], that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n$$

for $0 < x \leq 1$ by noting that the function g defined by $g(x) = f(1) - f(1-x)$ satisfies the hypotheses of the theorem. If there is a positive integer k such that $f^{(k)}(1) \neq 0$, then assume, without loss of generality, that k is the least such integer. Then $f \in A_\infty \subset A_{k+3}$ implies that $f^{(k+1)} \in K_0(k+1)$ if k is even, whereas $f^{(k+1)} \in K_1(k+1)$ if k is odd, from which it follows that $I(f^{(k+1)},k+1;) \in A_{k+3}$. Hence,

$$I(f^{(k+1)},k+1;) = I(f^{(k)},k;) - f^{(k)}(1)I(1,k;) = f - e(m,1,k;),$$

where $m = (-1)^{k-1}[1/(k!)]f^{(k)}(1) > 0$. If $f_1 = e(m,1,k;)$ and $f_2 = f - f_1$, then $f_1 \in A_\infty$ since $f_1 \in A_n$ for every n and $f_2 \in A_\infty$ since $f_2 \in A_{k+3}$ and

$$\nabla_h^n f_2(x) = \nabla_h^n [f(x) - e(m,1,k;x)] = \nabla_h^n f(x) \leq 0,$$

for $h > 0$, $[x,x+nh] \subset [0,1]$ and $n \geq k+3$. Since f_1 is not proportional to f, this gives a nonproportional decomposition of f, and f is therefore extremal. On the other hand, if $f^{(k)}(1) = 0$ for each positive integer k, then $f(x) = f(1)$ for $0 < x \leq 1$, and $f(0+) = f(0) = 0$ implies that $f = 0$.

The results of this chapter are summarized in the following theorem.

Theorem 2.1: The extremal elements of A_1 are the functions which assume exactly one positive value in $[0,1]$. The positive constant functions and the functions which are a positive constant on $(0,1]$ and zero at 0 are extremal elements of A_n, $n \geq 1$, and are therefore extremal in A_∞. The functions $e(m,\xi,n-1;x) = m[\xi^{n-1}-(\xi-x)^{n-1}]$, $x \varepsilon [0,\xi]$ and $m\xi^{n-1}$ for $x \varepsilon [\xi,1]$, where $m > 0$ and $0 < \xi \leq 1$, are extremal elements of A_n, $n \geq 2$. The only other extremal elements of A_n, $n \geq 3$, are those functions $e(m,1,k;)$, $1 \leq k \leq n-2$. The extremal elements of A_∞ which are continuous and zero at 0 are the functions $e(m,1,k;)$, $k \geq 1$.

Since A_n is a subcone of A_1 for $n > 1$, A_n is in the linear space A_1-A_1 and

$$A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots .$$

If $\varphi_1 : A_1-A_1 \to R$ such that $\varphi_1(f) = f(1)$, for $f \varepsilon A_1-A_1$, then it is easily seen that φ_1 is a linear functional and it follows that

$$H = [\varphi_1 : 1] = \{f : f \varepsilon A_1-A_1, f(1) = 1\}$$

is a hyperplane in A_1-A_1. Since H meets every ray of A_n in a unique point, $n \geq 1$, and does not contain the origin, that is the zero function, then the extreme points of $C_n = H \cap A_n$ are precisely those extremal elements f of A_n such that $f(1) = 1$.

Since $\{A_n\}$ is a nested sequence of cones, it follows that $\{C_n\} = \{H \cap A_n\}$ is a nested sequence of convex sets. If $f_0(0) = 0$ and $f_0(x) = 1$, $x \varepsilon (0,1]$, and $f_1(x) = 1$ for $x \varepsilon [0,1]$, then f_0 and f_1 are extreme points of C_n for $n \geq 1$. In fact,

$$\text{ext } C_1 = \{f : f \in A_1 \text{ and } f(x) \text{ is either } 0 \text{ or } 1 \text{ for } x \in [0,1]\},$$

$$\text{ext } C_2 = \{f_0, f_1\} \cup \{e((1/\xi),\xi,1;) : 0 < \xi \leq 1\},$$

$$\text{ext } C_{n+1} = \{f_0, f_1\} \cup \{e(1,1,k;) : 1 \leq k \leq n-1\} \cup \{e((1/\xi)^n,\xi,n;) : 0 < \xi \leq 1\}$$

for $n > 1$, and

$$\text{ext } C_\infty = \{f_0, f_1\} \cup \{e(1,1,k;) : k \geq 1\},$$

where $C_\infty = \cap C_n$. Hence, ext C_n is uncountable, $n \geq 1$, and ext C_∞ is countable.

Figure 2.1 gives a pictorial representation of C_n, $1 \leq n \leq 5$, and illustrates how C_{n+1} is related to C_n. Each region C_n is bounded below by the positive x-axis. C_1 is bounded above by the semicircle with radius 2 and center $(2,0)$. The curved portion of the boundary of C_2 is a part of the semicircle with radius $1+(1/2)$ and center $(1+(1/2),0)$ and the line segment with endpoints f_1 and $e(1,1,1;)$ is tangent to this semicircle at $e(1,1,1;)$; the curved portion of the boundary of C_3 is a part of the semicircle with radius $1+(1/4)$ and center $(1+(1/4),0)$ and the line segment with endpoints $e(1,1,1;)$ and $e(1,1,2;)$ is tangent to this semicircle at $e(1,1,2;)$; and so forth. This diagram should aid in understanding the distribution of the extremal elements of A_n.

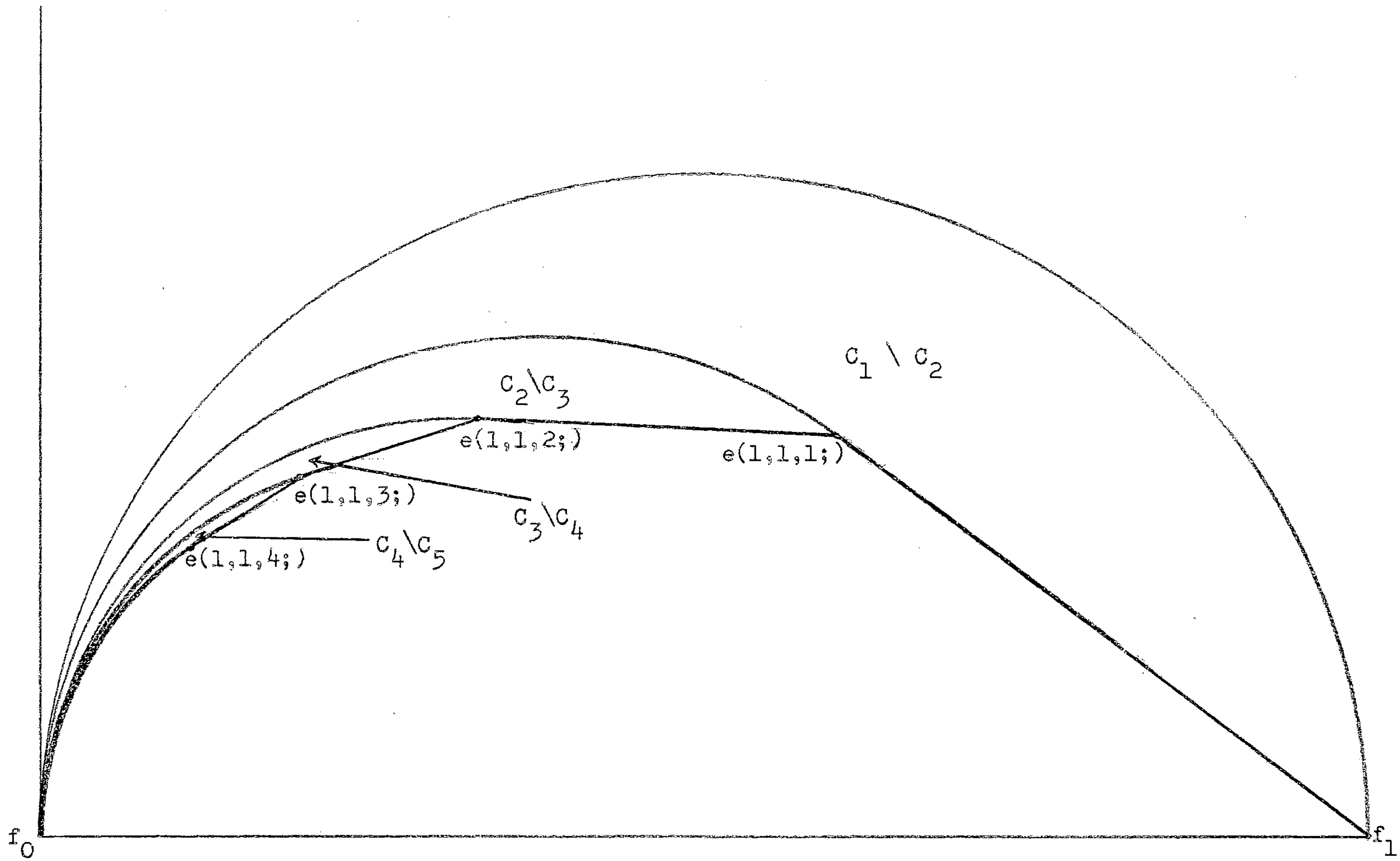

Figure 2.1.

CHAPTER III

EXTREMAL ELEMENTS OF THE CONVEX CONE OF n-MONOTONE FUNCTIONS

Let $C(i_0, i_1)$, i_0 and $i_1 = 0$ or 1, be the set of real functions f on $[0,1]$ such that $(-1)^{(i_0)} f$ is nonnegative and

$$(-1)^{(i_1)} \Delta_h^1 f(x) = (-1)^{(i_1)} [f(x+h) - f(x)] \geq 0,$$

$h > 0$, for $[x, x+h] \subset [0,1]$. Let $C(i_0, i_1, \ldots, i_n)$, $n > 1$, be the set of functions belonging to $C(i_0, \ldots, i_{n-1})$ such that

$$(-1)^{(i_n)} \Delta_h^n f(x) = (-1)^{(i_n)} [\Delta_h^{n-1} f(x+h) - \Delta_h^{n-1} f(x)] \geq 0$$

for $[x, x+nh] \subset [0,1]$, where $i_k = 0$ or $i_k = 1$ for $0 \leq k \leq n$. If $f \in C(i_0, \ldots, i_n)$, then f is said to be an <u>n-monotone function</u>. Since the sum of two n-monotone functions is in $C(i_0, \ldots, i_n)$ and since a non-negative real multiple of an n-monotone function is an n-monotone function, the set of n-monotone functions forms a convex cone. It is the purpose of this chapter to determine the extremal elements of $C(i_0, \ldots, i_n)$, $n \geq 1$. It should be noted at this point that in order to find the extremal elements of $C(i_0, \ldots, i_n)$, $n \geq 1$, it is sufficient to determine the extremal elements of $C(0, i_1, \ldots, i_n)$, since

$$C(1, i_1, \ldots, i_n) = -C(0, 1-i_1, \ldots, 1-i_n).$$

Proposition 3.1: The extremal elements of $C(0,i_1)$ are the functions in $C(0,i_1)$ which assume exactly one positive value in $[0,1]$.

Proof: If $i_1 = 0$ then $C(0,0) = A_1$, and the proposition is true by Proposition 2.1. If $i_1 = 1$ then $f \varepsilon C(0,1)$ if, and only if, $g \varepsilon C(0,0) = A_1$, where $g(x) = f(1-x)$ for $x \varepsilon [0,1]$. It then follows that f is an extremal element of $C(0,1)$ if, and only if, g is an extremal element of A_1. Thus, f is extremal in $C(0,1)$ if, and only if, f assumes exactly one positive value in $[0,1]$.

Extremal Elements of $C(i_0,i_1,i_2)$

Let $f \varepsilon C(0,i_1,i_2)$ and let $a_0 = 0$ if $i_1 = 0$ and $a_0 = 1$ if $i_1 = 1$. If $f(a_0) > 0$ and f is not constant, then take $f_1 = f(a_0)$ and $f_2 = f - f_1$. In so doing, f_1 and $f_2 \varepsilon C(0,i_1,i_2)$ and f_1 and f_2 are not proportional to f. Since the same technique still can be used for $C(0,i_1,\ldots,i_n)$, $n > 2$, the only extremal elements of $C(0,i_1,\ldots,i_n)$ such that $f(a_0) > 0$ are the positive constant functions.

Let $f \varepsilon C(0,i_1,i_2)$ and define

$$a_0' = (1/2) + (-1)^{(i_2)}[(1/2) - a_0],$$

where a_0 is defined as above. It has already been noted that an A_2 function must be continuous on $(0,1]$; in this case, $A_2 = C(0,0,1)$ and $a_0' = a_0 = 0$. By similar reasoning, it can be shown that if $f \varepsilon C(0,i_1,i_2)$, then f must be continuous on $[0,1]$ except at a_0'. It follows that the only extremal elements of $C(0,i_1)$ that are in $C(0,i_1,i_2)$ are those which are continuous on $[0,1]$ except, possibly, at a_0', and these functions are again extremal in $C(0,i_1,i_2)$.

If $f \in C(0,i_1,0)$, f is not constant on (0,1) and f is discontinuous at $a_0' = 1-a_0$, then take $f_1(x) = 0$ for $x \in [0,1]$ and $x \neq a_0'$,

$$f_1(a_0') = f(a_0') - \lim_{x \to a_0'} f(x) > 0$$

and $f_2 = f - f_1$. In so doing, f_1 and $f_2 \in C(0,i_1,0)$ and f_1 and f_2 are not proportional to f. Since the same technique still can be used for $C(0,i_1,0,\ldots,i_n)$, $n > 2$, the only extremal elements of $C(0,i_1,0,\ldots,i_n)$ that are discontinuous at $a_0' = 1-a_0$ are the functions which are positive at a_0' and zero elsewhere on [0,1].

On the other hand, if $f \in C(0,i_1,1)$, f is not constant on (0,1) and f is discontinuous at $a_0' = a_0$, then take

$$f_1(x) = \lim_{x \to a_0'} f(x) > 0,$$

$x \in [0,1]$ and $x \neq a_0'$, $f_1(a_0') = 0$ and $f_2 = f - f_1$. Then f_1 and f_2 are in $C(0,i_1,1)$ and f_1 and f_2 are not proportional to f. Again, since the same technique can be used for $C(0,i_1,1,\ldots,i_n)$, $n > 2$, the only extremal elements of $C(0,i_1,1,\ldots,i_n)$ that are discontinuous at $a_0' = a_0$ are the functions which are zero at a_0' and equal to a positive constant elsewhere on [0,1].

Consequently, the extremal elements of $C(0,i_1,\ldots,i_n)$, $n > 1$, which are not extremal in $C(0,i_1)$ must be zero at a_0 and continuous on [0,1]. The extremal elements of C(0,1,1) which are extremal in C(0,1) are the positive constant functions and the functions f such that $f = c > 0$ on [0,1) while $f(1) = 0$. The remaining extremal elements of C(0,1,1) are found in the following proposition by using the fact that the extremal elements of $A_2 = C(0,0,1)$ are known.

Proposition 3.2: Let $f \in C(0,1,1)$ such that $f(1) = 0$, $f \neq 0$ and f is continuous at 1. Then f is an extremal element of $C(0,1,1)$ if, and only if, $f(x) = m(1-\xi)$, $x \in [0,\xi]$ and $m(1-x)$ for $x \in [\xi,1]$, where $0 \leq \xi < 1$ and $m > 0$.

Proof: It is easily seen that $f \in C(0,1,1)$ if, and only if, $g \in C(0,0,1) = A_2$, where $g(x) = f(1-x)$, $x \in [0,1]$, which implies that f is an extremal element of $C(0,1,1)$ if, and only if, g is extremal in A_2. Therefore, since f is continuous at 1 and nonconstant, f is extremal in $C(0,1,1)$ if, and only if, $f(1-x) = mx$, $x \in [0,\xi]$ and $m\xi$ for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$. It follows that $f(x) = m\xi$, $x \in [0,1-\xi]$ and $m(1-x)$ for $x \in [1-\xi,1]$, which is equivalent to $f(x) = m(1-\beta)$, $x \in [0,\beta]$ and $m(1-x)$ for $x \in [\beta,1]$, where $0 \leq \beta < 1$ and $m > 0$.

McLachlan [8] has found the extremal elements of the convex cone B_2 which is a subcone of $C(0,0,0)$. In fact, B_2 is the set of functions in $C(0,0,0)$ which are continuous at 1. It easily follows from McLachlan's results that the extremal elements of $C(0,0,0)$ which are continuous at 1 and nonconstant are precisely those functions f such that $f(x) = 0$, $x \in [0,\xi]$ and $m(x-\xi)$ for $x \in [\xi,1]$, where $0 \leq \xi < 1$ and $m > 0$. Once this is known, the extremal elements of $C(0,1,0)$ which are nonconstant and continuous at 0 can be determined.

Proposition 3.3: Let $f \in C(0,1,0)$ such that $f(1) = 0$, $f \neq 0$ and f is continuous at 0. Then f is an extremal element of $C(0,1,0)$ if, and only if, $f(x) = m(\xi-x)$, $x \in [0,\xi]$ and 0 for $x \in [\xi,1]$, where $0 < \xi \leq 1$ and $m > 0$.

Proof: It is easily seen that $f \in C(0,1,0)$ if, and only if, $g \in C(0,0,0)$, where $g(x) = f(1-x)$, $x \in [0,1]$, which implies f is an extremal element of $C(0,1,0)$ if, and only if, g is extremal in $C(0,0,0)$. Therefore, since f is continuous at 0 and nonconstant, f is extremal in $C(0,1,0)$ if, and only if, $f(1-x) = 0$, $x \in [0,\xi]$ and $m(x-\xi)$ for $x \in [\xi,1]$, where $0 \leq \xi < 1$ and $m > 0$. It follows that $f(x) = m(1-\xi-x)$, $x \in [0,1-\xi]$ and 0 for $x \in [1-\xi,1]$, which is equivalent to $f(x) = m(\beta-x)$, $x \in [0,\beta]$ and 0 for $x \in [\beta,1]$, where $0 < \beta \leq 1$ and $m > 0$.

By using the results from Chapter II, McLachlan's results and Propositions 3.2 and 3.3, it can be verified that any extremal element of $C(0,i_1,i_2)$ which is nonconstant and continuous on $[0,1]$ must be of the form

$$m[h(x,\xi;i_1,i_2)+h(\xi,\xi;i_1,i_2)+(-1)^{(i_2)}|h(x,\xi;i_1,i_2)-h(\xi,\xi;i_1,i_2)|], \quad (3.1)$$

where $m > 0$ and

$$h(x,\xi;i_1,i_2) = (1/2) - (-1)^{(i_1)}[(1/2) - x]$$
$$- (1/2)[1 + (-1)^{(i_2)}]\{(1/2) - (-1)^{(i_1)}[(1/2) - \xi]\}$$

for $x \in [0,1]$ and $\xi \in (0,1)$ or $\xi = a_1 = (1/2)[1 - (-1)^{(i_1+i_2)}]$. Let $mf(\xi,i_1,i_2;)$ denote the function such that $mf(\xi,i_1,i_2;x)$ is given by (3.1). The results to this point are summarized in the following theorem.

<u>Theorem 3.1</u>: The extremal elements of $C(0,i_1)$ are the functions which assume exactly one positive value on $[0,1]$. The extremal elements of $C(0,i_1)$ which are continuous on $[0,1]$ except, possibly, at $a_0' = (1/2)[1 + (-1)^{(i_1+i_2)}]$ are again extremal in $C(0,i_1,i_2)$. The

remaining extremal elements of $C(0,i_1,i_2)$ are the functions $mf(\xi,i_1,i_2;)$, where $m > 0$ and $\xi \varepsilon (0,1)$ or $\xi = a_1$ for

$$a_1 = (1/2)[1 - (-1)^{(i_1+i_2)}].$$

Extremal Elements of $C(i_0,i_1,\ldots,i_n)$, $n > 2$

The extremal elements of the convex cone $C(i_0,i_1,\ldots,i_n)$, $n > 2$, will be found by integrating the extremal elements of a convex cone which is similar to $C(i_{n-2},i_{n-1},i_n)$. It is necessary to undertake a study of the properties of the derivatives of an n-monotone function before defining this cone. The following two lemmas should be compared with Lemma 2.3.

Lemma 3.1: If $f \varepsilon C(i_0,i_1,\ldots,i_{n+2})$, $n > 0$, then $f^{(n)}$ exists on $(0,1)$ and

$$(-1)^{(i_{n+k})}\Delta_h^k f^{(n)}(x) \geq 0$$

for $h > 0$, $[x,x+kh] \subset (0,1)$ and $k = 0,1,2$, where $\Delta_h^0 f^{(n)}(x) = f^{(n)}(x)$.

Proof: Since $\Delta_h^{n+2} (-1)^{(i_{n+2})}f(x) = (-1)^{(i_{n+2})}\Delta_h^{n+2} f(x) \geq 0$ for $h > 0$ and $[x,x+(n+2)h] \subset [0,1]$, it follows that $f^{(n)}$ exists on $(0,1)$ and $(-1)^{(i_{n+2})}f^{(n)}$ is convex on $(0,1)$ [2]. Thus,

$$(-1)^{(i_{n+2})}\Delta_h^2 f^{(n)}(x) = \Delta_h^2 (-1)^{(i_{n+2})}f^{(n)}(x) \geq 0$$

for $h > 0$, $[x,x+2h] \subset (0,1)$. The proof that $(-1)^{(i_n)}f^{(n)}(x) \geq 0$, $x \varepsilon (0,1)$ and $(-1)^{(i_{n+1})}\Delta_h^1 f^{(n)}(x) \geq 0$, $[x,x+h] \subset (0,1)$ will be by induction on n.

If $f \in C(i_0,i_1,i_2,i_3)$, then f' exists on $(0,1)$ and $(-1)^{(i_k)}\Delta_h^k f(x) \geq 0$ for $[x,x+kh] \subset [0,1]$ and $k = 1,2$. It follows that

$$(-1)^{(i_k)}\Delta_h^{k-1}\ \Delta_\delta^1\ f(x) \geq 0$$

for $h > 0$, $\delta > 0$, $[x,x+(k-1)h+\delta] \subset [0,1]$ (cf. Proposition 1.1). Hence,

$$(-1)^{(i_k)}\Delta_h^{k-1}\ \frac{f(x+\delta) - f(x)}{\delta} \geq 0,$$

which implies that

$$(-1)^{(i_k)}\Delta_h^{k-1}\ f'(x) \geq 0$$

for $k = 1,2$, and thus, the lemma is true for $n = 1$.

Assume that if $f \in C(i_0,i_1,\ldots,i_{n+2})$, $n > 1$, then $(-1)^{(i_{n+k})}\Delta_h^k\ f^{(n)}(x) \geq 0$ for $h > 0$, $[x,x+kh] \subset (0,1)$ and $k = 0,1$. If $f \in C(i_0,i_1,\ldots,i_{n+2},i_{n+3})$, then $f \in C(i_0,i_1,\ldots,i_{n+2})$, and it follows from the induction hypothesis and the first part of the proof that

$$(-1)^{(i_{n+k})}\Delta_h^{k-1}\ \Delta_\delta^1\ f^{(n)}(x) \geq 0$$

for $h > 0$, $\delta > 0$, $[x,x+(k-1)h+\delta] \subset (0,1)$ and $k = 1,2$. Hence,

$$(-1)^{(i_{n+k})}\Delta_h^{k-1}\ \frac{f^{(n)}(x+\delta) - f^{(n)}(x)}{\delta} \geq 0,$$

which, by replacing k with k+1, implies

$$(-1)^{(i_{n+1+k})}\Delta_h^k\ f^{(n+1)}(x) \geq 0$$

for $k = 0,1$. This completes the induction.

The following definition is included here in order to simplify the notation in the proofs that follow.

Definition 3.1: If g is a real continuous function on (0,1), then define the function

$$I(g,i_k,i_{k+1},\ldots,i_m,i_{m+1};)$$

by the equation

$$I(g,i_k,i_{k+1},\ldots,i_m,i_{m+1};x)$$

$$= \int_{a_k}^{x} \int_{a_{k+1}}^{t_{k+1}} \cdots \int_{a_{m-1}}^{t_{m-1}} \int_{a_m}^{t_m} g(t)\ dt\ dt_m \ \ldots\ dt_{k+1}$$

for $x \in (0,1)$ and $a_j = (1/2)[1 - (-1)^{(i_j+i_{j+1})}]$, where $i_j = 0$ or $i_j = 1$, $0 \leq k \leq j \leq m$.

Lemma 3.2: If $f \in C(i_0,i_1,\ldots,i_{n+2})$, $n > 0$, then

$$\lim_{x \to 1-a_0} I(f^{(n)},i_0,i_1,\ldots,i_{n-1},i_n;x)$$

exists and is finite.

Proof: The proof will be by induction on n. If $f \in C(i_0,i_1,i_2,i_3)$, then

$$I(f',i_0,i_1;x) = \int_{a_0}^{x} f'(t)\ dt = f(x) - f(a_0),$$

which implies that

$$\lim_{x \to 1-a_0} I(f', i_0, i_1; x) = f(1-a_0) - f(a_0),$$

and the lemma is true for $n = 1$.

Assume the lemma is true for $n > 1$ and let f be in $C(i_0, i_1, \ldots, i_{n+2}, i_{n+3})$. Then $f \varepsilon C(i_0, i_1, \ldots, i_{n+2})$, and by Lemma 3.1,

$$(-1)^{(i_{n+k})} \Delta_h^k f^{(n)}(x) \geq 0$$

for $[x, x+kh] \subset (0,1)$ and $k = 0, 1$. If $i_n = i_{n+1}$, then $a_n = 0$ and

$$f^{(n)}(0) = f^{(n)}(0+) = \lim_{x \to 0+} f^{(n)}(x)$$

is finite. If $i_n \neq i_{n+1}$, then $a_n = 1$ and

$$f^{(n)}(1) = f^{(n)}(1-) = \lim_{x \to 1-} f^{(n)}(x)$$

is finite. It follows that

$$f^{(n)}(a_n) = \lim_{x \to a_n} f^{(n)}(x)$$

is finite. Therefore,

$$\lim_{x \to 1-a_0} I(f^{(n+1)}, i_0, i_1, \ldots, i_n, i_{n+1}; x)$$

$$= \lim_{x \to 1-a_0} I(f^{(n)} - f^{(n)}(a_n), i_0, i_1, \ldots, i_n; x)$$

$$= \lim_{x \to 1-a_0} I(f^{(n)}, i_0, i_1, \ldots, i_n; x) - f^{(n)}(a_n) I(1, i_0, i_1, \ldots, i_n; 1-a_0)$$

exists and is finite by the induction hypothesis.

It is a consequence of Lemmas 3.1 and 3.2 that if $f \varepsilon C(i_0, \ldots, i_n)$, $n > 2$, then $f^{(n-2)}$ is an element of the convex cone given in the following definition.

Definition 3.2: Let $K(i_{n-2}, i_{n-1}, i_n)$ be the set of real functions g on (0,1) such that

$$(-1)^{(i_{n-2+k})}\Delta_h^k\, g(x) \geq 0$$

for $h > 0$, $[x, x+kh] \subset (0,1)$, where $k = 0, 1, 2$, and

$$\underset{x \to 1-a_0}{\text{limit}}\; I(g, i_0, i_1, \ldots, i_{n-2}; x)$$

is finite.

From this point, $I(g, i_0, i_1, \ldots, i_{n-2};)$ will denote the function which is the continuous extension to [0,1] of the function given in Definition 3.1, where $g \varepsilon K(i_{n-2}, i_{n-1}, i_n)$. The proof of the following proposition is very similar to that given for Lemma 2.1 and is not included here.

Proposition 3.4: A function $g \varepsilon K(i_{n-2}, i_{n-1}, i_n)$ is an extremal element of $K(i_{n-2}, i_{n-1}, i_n)$ if, and only if, $g \not\equiv 0$ and $g = f|(0,1)$, the restriction of f to (0,1), where f is an extremal element of $C(i_{n-2}, i_{n-1}, i_n)$.

The result obtained here will be used in the proof of the next lemma.

Lemma 3.3: If g is a real continuous function on (0,1) such that $I(g, i_0, i_1, \ldots, i_n; x)$ is finite on (0,1) and $(-1)^{(i_n)} g(x) \geq 0$, $x \varepsilon (0,1)$, then

$$(-1)^{(i_k)} I(g, i_k, i_{k+1}, \ldots, i_n; x) \geq 0$$

for $x \varepsilon (0,1)$ and $0 \leq k \leq n-1$.

Proof: The proof will be by finite induction on k beginning with $k = n-1$. If $i_{n-1} = i_n$, then $a_{n-1} = 0$ and

$$(-1)^{(i_{n-1})} I(g, i_{n-1}, i_n; x) = (-1)^{(i_{n-1})} \int_0^x g(t)\,dt = \int_0^x (-1)^{(i_n)} g(t)\,dt \geq 0,$$

whereas if $i_{n-1} \neq i_n$, then $a_{n-1} = 1$ and

$$(-1)^{(i_{n-1})} I(g, i_{n-1}, i_n; x) = (-1)^{(i_{n-1})} \int_1^x g(t)\,dt = -\int_1^x (-1)^{(i_n)} g(t)\,dt \geq 0,$$

for $x \varepsilon (0,1)$, since $(-1)^{(i_n)} g(t) \geq 0$, $t \varepsilon (0,1)$.

Assume that $(-1)^{(i_k)} I(g, i_k, i_{k+1}, \ldots, i_n; x) \geq 0$ for $x \varepsilon (0,1)$, where $0 < k < n-1$. If $i_{k-1} = i_k$, then $a_{k-1} = 0$ and

$$(-1)^{(i_{k-1})} I(g, i_{k-1}, i_k, \ldots, i_n; x) = (-1)^{(i_{k-1})} \int_0^x I(g, i_k, \ldots, i_n; t)\,dt$$

$$= \int_0^x (-1)^{(i_k)} I(g, i_k, \ldots, i_n; t)\,dt \geq 0,$$

by the induction hypothesis. On the other hand, if $i_{k-1} \neq i_k$, then $a_{k-1} = 1$ and

$$(-1)^{(i_{k-1})} I(g,i_{k-1},i_k,\ldots,i_n;x) = (-1)^{(i_{k-1})} \int_1^x I(g,i_k,\ldots,i_n;t)\, dt$$

$$= -\int_1^x (-1)^{(i_k)} I(g,i_k,\ldots,i_n;t)\, dt \geq 0,$$

again by the induction hypothesis.

In the next lemma, an n-monotone function f is obtained by specifying $f^{(n-2)}$ on $(0,1)$.

<u>Lemma 3.4</u>: If $g \in K(i_{n-2},i_{n-1},i_n)$, $n > 2$, then $f \in C(i_0,i_1,\ldots,i_n)$, where $f = I(g,i_0,i_1,\ldots,i_{n-2};)$.

Proof: The technique used in the proof of Lemma 3.2 can be employed here to show that f is finite on $[0,1]$. Since, by Lemma 3.3,

$$(-1)^{(i_0)} f(x) = (-1)^{(i_0)} I(g,i_0,i_1,\ldots,i_{n-2};x) \geq 0$$

for $x \in (0,1)$, $f(a_0) = 0$ and f is continuous at $1-a_0$, it follows that $(-1)^{(i_0)} f(x) \geq 0$ for $x \in [0,1]$. The proof that $(-1)^{(i_k)} \Delta_h^k f(x) \geq 0$ for $h > 0$, $[x,x+kh] \subset [0,1]$, where $1 \leq k \leq n-3$, will be by induction on k.

For $h > 0$ and $[x,x+h] \subset [0,1]$,

$$\Delta_h^1 f(x) = \int_x^{x+h} I(g,i_1,\ldots,i_{n-2};t)\, dt = h\, I(g,i_1,\ldots,i_{n-2};\xi_1),$$

where $x < \xi_1 < x+h$. Assume that $\Delta_h^k f(x) = h^k I(g,i_k,i_{k+1},\ldots,i_{n-2};\xi_k)$, where $0 \leq x < \xi_k < x+kh \leq 1$ and $1 < k < n-3$. For $h > 0$ and

$[x, x+(k+1)h] \subset [0,1]$,

$$\Delta_h^{k+1} f(x) = \Delta_h^1 \Delta_h^k f(x)$$

$$= \Delta_h^1 h^k I(g, i_k, i_{k+1}, \ldots, i_{n-2}; \xi_k)$$

$$= h^k \int_{\xi_k}^{\xi_k+h} I(g, i_{k+1}, \ldots, i_{n-2}; t)\, dt$$

$$= h^{k+1} I(g, i_{k+1}, \ldots, i_{n-2}; \xi_{k+1}),$$

where $\xi_k < \xi_{k+1} < \xi_k+h$ and hence, $x < \xi_{k+1} < x+(k+1)h$. Therefore, it follows from Lemma 3.3 that

$$(-1)^{(i_k)}\Delta_h^k f(x) = (-1)^{(i_k)} h^k I(g, i_k, \ldots, i_{n-2}; \xi_k) \geq 0,$$

where $0 \leq x < \xi_k < x+kh \leq 1$ and $1 \leq k \leq n-3$.

If $h > 0$ and $[x, x+kh] \subset [0,1]$, for $n-2 \leq k \leq n$, then

$$\Delta_h^k f(x) = \Delta_h^{k-n+3} \Delta_h^{n-3} f(x)$$

$$= \Delta_h^{k-n+3} h^{n-3} I(g, i_{n-3}, i_{n-2}; \xi_{n-3})$$

$$= \Delta_h^{k-n+2} \Delta_h^1 h^{n-3} I(g, i_{n-3}, i_{n-2}; \xi_{n-3})$$

$$= h^{n-3}\Delta_h^{k-n+2} \int_{\xi_{n-3}}^{\xi_{n-3}+h} g(t)\, dt$$

$$= h^{n-2}\Delta_h^{k-n+2} g(\xi),$$

where $x < \xi < x+(n-2)h$. It easily follows that $(-1)^{(i_k)}\Delta_h^k f(x) \geq 0$

for $h > 0$, $[x,x+kh] \subset [0,1]$, where $n-2 \leq k \leq n$, and thus, $f \in C(i_0,i_1,\ldots,i_n)$.

In the proofs that follow, $f^{(k)}(a_k)$ should be interpreted as

$$f^{(k)}(a_k) = \lim_{x \to a_k} f^{(k)}(x),$$

where $f \in C(i_0,i_1,\ldots,i_n)$ and $1 \leq k \leq n-2$. Since $f^{(k)} \in K(i_k,i_{k+1},i_{k+2})$, this limit will always exist and be finite. It is shown in the following proposition that extremal elements of $C(i_0,i_1,\ldots,i_n)$ can be obtained directly from extremal elements of $K(i_{n-2},i_{n-1},i_n)$.

Proposition 3.5: Let $g \in K(i_{n-2},i_{n-1},i_n)$, $n > 2$, and let $f = I(g,i_0,i_1,\ldots,i_{n-2};)$. If g is an extremal element of $K(i_{n-2},i_{n-1},i_n)$, then f is an extremal element in $C(i_0,i_1,\ldots,i_n)$.

Proof: It has already been shown that $f \in C(i_0,i_1,\ldots,i_n)$ (cf. Lemma 3.4). If f_1 and $f_2 \in C(i_0,i_1,\ldots,i_n)$ such that $f = f_1 + f_2$, then $f_1^{(n-2)}$ and $f_2^{(n-2)} \in K(i_{n-2},i_{n-1},i_n)$ and $f_1^{(n-2)} + f_2^{(n-2)} = f^{(n-2)} = g$. This implies that there are constants $\lambda_i \geq 0$, $i = 1,2$ such that $f_i^{(n-2)} = \lambda_i g$. It is evident from the definition of f that $f^{(k)}(a_k) = 0$, where $0 \leq k \leq n-3$. This, together with the fact that $f_i^{(k)}$ is in $K(i_k,i_{k+1},i_{k+2})$ for $0 \leq k \leq n-3$, implies that $f_i^{(k)}(a_k) = 0$, $i = 1,2$. Hence,

$$\begin{aligned} f_i &= I(f_i^{(n-2)},i_0,i_1,\ldots,i_{n-2};) \\ &= I(\lambda_i g,i_0,i_1,\ldots,i_{n-2};) \\ &= \lambda_i I(g,i_0,i_1,\ldots,i_{n-2};) \\ &= \lambda_i f, \end{aligned}$$

for $i = 1, 2$, and f is therefore extremal.

Thus, Proposition 3.5 gives a sufficient condition for f to be extremal in $C(i_0, i_1, \ldots, i_n)$. The following lemma will be used in the proof of Proposition 3.6, in which a necessary condition for f to be an extremal element in $C(i_0, i_1, \ldots, i_n)$ is given.

Lemma 3.5: Let $f \varepsilon C(i_0, i_1, \ldots, i_n)$, $n > 2$, such that f is not an extremal element of $C(i_0, i_1, \ldots, i_{n-1})$. If f is extremal in $C(i_0, i_1, \ldots, i_n)$, then $f^{(k)}(a_k) = 0$ for $0 \leq k \leq n-3$.

Proof: Since f is not extremal in $C(i_0, i_1)$, it follows that $f(a_0) = 0$. The proof that $f^{(k)}(a_k) = 0$ for $1 \leq k \leq n-3$ will be by contraposition.

Suppose there is a k such that $1 \leq k \leq n-3$ and $f^{(k)}(a_k) \neq 0$. Let p denote the smallest such integer. Since $f \varepsilon C(i_0, i_1, \ldots, i_n) \subset C(i_0, i_1, \ldots, i_{p+3})$, it follows that $f^{(p+1)} \varepsilon K(i_{p+1}, i_{p+2}, i_{p+3})$ and

$$\begin{aligned} I(f^{(p+1)}, i_0, i_1, \ldots, i_{p+1}; x) &= I(f^{(p)} - f^{(p)}(a_p), i_0, i_1, \ldots, i_p; x) \\ &= I(f^{(p)}, i_0, i_1, \ldots, i_p; x) - I(f^{(p)}(a_p), i_0, i_1, \ldots, i_p; x) \\ &= f(x) - I(f^{(p)}(a_p), i_0, i_1, \ldots, i_p; x) \qquad (3.2) \end{aligned}$$

for $x \varepsilon [0,1]$, because $f^{(k)}(a_k) = 0$ for $0 \leq k \leq p-1$. Since $f^{(p)}$ is in $K(i_p, i_{p+1}, i_{p+2})$ and $f^{(p)}(a_p) \neq 0$, the constant function $f^{(p)}(a_p)$ is extremal in $K(i_p, i_{p+1}, i_{p+2})$ by Proposition 3.4. If

$$f_1 = I(f^{(p)}(a_p), i_0, i_1, \ldots, i_p;),$$

then by Proposition 3.5, f_1 is extremal in $C(i_0, i_1, \ldots, i_{p+2})$. It was shown in the proof of Lemma 3.4 that $\Delta_h^{p+1} f_1(x) = h^p \Delta_h^1 f^{(p)}(a_p) = 0$ for $h > 0$ and $[x, x+(p+1)h] \subset [0,1]$. It follows that $\Delta_h^k f_1(x) = 0$ for $h > 0$, $[x, x+kh] \subset [0,1]$ and $p+2 \leq k \leq n$, which implies that $f_1 \in C(i_0, \ldots, i_n)$. If $f_2 = f - f_1$, then

$$f_2 = I(f^{(p+1)}, i_0, i_1, \ldots, i_{p+1}),$$

by equation 3.2. Hence, by Lemmas 3.1, 3.2 and 3.4, $f_2 \in C(i_0, \ldots, i_{p+3})$, and since

$$(-1)^{(i_k)}\Delta_h^k f_2(x) = (-1)^{(i_k)}[\Delta_h^k f(x) - \Delta_h^k f_1(x)] = (-1)^{(i_k)}\Delta_h^k f(x) \geq 0$$

for $h > 0$, $[x, x+kh] \subset [0,1]$ and $p+3 \leq k \leq n$, it follows that f_2 is in $C(i_0, i_1, \ldots, i_n)$. Thus, f_1 and $f_2 \in C(i_0, i_1, \ldots, i_n)$ such that $f = f_1 + f_2$, and since f_1 is extremal in $C(i_0, i_1, \ldots, i_{p+2})$, $1 \leq p \leq n-3$, f_1 is not proportional to f. Therefore, f is not extremal in $C(i_0, i_1, \ldots, i_n)$. It follows, by contraposition, that if f is extremal in $C(i_0, i_1, \ldots, i_n)$, then $f^{(k)}(a_k) = 0$ for $1 \leq k \leq n-3$.

<u>Proposition 3.6</u>: Let $f \in C(i_0, i_1, \ldots, i_n)$, $n > 2$, such that f is not an extremal element of $C(i_0, i_1, \ldots, i_{n-1})$. If f is extremal in $C(i_0, i_1, \ldots, i_n)$, then $f^{(n-2)}$ is nonconstant and $f^{(n-2)}$ is extremal in $K(i_{n-2}, i_{n-1}, i_n)$.

Proof: By Lemma 3.5, $f^{(k)}(a_k) = 0$ for $0 \leq k \leq n-3$. If g_1 and g_2 are in $K(i_{n-2}, i_{n-1}, i_n)$ such that $f^{(n-2)} = g_1 + g_2$, then by Lemma 3.4, f_1 and $f_2 \in C(i_0, i_1, \ldots, i_n)$ such that $f = f_1 + f_2$, where $f_i = I(g_i, i_0, i_1, \ldots, i_{n-2};)$, $i = 1,2$. Since f is extremal in $C(i_0, i_1, \ldots, i_n)$, there are constants $\lambda_i \geq 0$ such that $f_i = \lambda_i f$, $i = 1,2$, which implies that $g_i = f_i^{(n-2)} = \lambda_i f^{(n-2)}$, $i = 1,2$, and $f^{(n-2)}$ is therefore extremal.

If $f^{(n-2)}$ is constant, then $f^{(n-2)} \neq 0$ and $f^{(n-3)}$ is linear on $(0,1)$. If $n = 3$, then f is extremal in $C(i_0, i_1, i_2)$, since f is linear on $(0,1)$ and $f(a_0) = 0$. If $n > 3$, then $f^{(n-3)}$ is extremal in $K(i_{n-3}, i_{n-2}, i_{n-1})$, since $f^{(n-3)}$ is linear on $(0,1)$ and $f^{(n-3)}(a_{n-3}) = 0$. It follows from Proposition 3.5 that f is extremal in $C(i_0, i_1, \ldots, i_{n-1})$, since $f = I(f^{(n-3)}, i_0, \ldots, i_{n-3};)$. However, since f is not an extremal element of $C(i_0, i_1, \ldots, i_{n-1})$, it must follow that $f^{(n-2)}$ is nonconstant.

If $f \varepsilon C(i_0, i_1, \ldots, i_n)$, $n > 2$, such that f is an extremal element of $C(i_0, \ldots, i_{n-1})$, then since $C(i_0, i_1, \ldots, i_n)$ is a subcone of $C(i_0, \ldots, i_{n-1})$, f is extremal in $C(i_0, i_1, \ldots, i_n)$. This set of functions is the subject of the following proposition.

<u>Proposition 3.7</u>: Let $f \varepsilon C(i_0, i_1, \ldots, i_n)$, $n > 2$, such that f is nonconstant and continuous on $[0,1]$. If f is an extremal element of $C(i_0, \ldots, i_{n-1})$, then f is linear on $[0,1]$ and $f(a_0) = 0$, or there is an integer k, $3 \leq k \leq n-1$, such that $f^{(k-2)}$ is extremal in $K(i_{k-2}, i_{k-1}, i_k)$, $f^{(p)}(a_p) = 0$ for $0 \leq p \leq k-3$ and $f^{(k-1)}$ is constant on $(0,1)$.

Proof: Since $f^{(n-2)}$ is continuous on $(0,1)$, by Lemma 3.1, $f^{(p)}$ is continuous on $(0,1)$ for $1 \leq p \leq n-2$. Since f is a nonconstant extremal element of $C(i_0, i_1, \ldots, i_n)$, it follows that $f(a_0) = 0$. There is a unique integer k, $2 \leq k \leq n-1$, such that f is extremal in $C(i_0, \ldots, i_k)$ but is not extremal in $C(i_0, \ldots, i_{k-1})$. If $k = 2$, then f is a non-constant extremal element of $C(i_0, i_1, i_2)$, and it follows that $(-1)^{(i_1)} f'$ assumes exactly one positive value in $(0,1)$ (cf. Theorem 3.1). Since

f' must be continuous on $(0,1)$, $(-1)^{(i_1)}f' = c > 0$ on $(0,1)$; that is, f is linear on $[0,1]$.

If $k > 2$, then by Lemma 3.5, $f^{(p)}(a_p) = 0$ for $0 \leq p \leq k-3$, and by Proposition 3.6, $f^{(k-2)}$ is extremal in $K(i_{k-2}, i_{k-1}, i_k)$. It follows from Proposition 3.4 and Theorem 3.1 that $(-1)^{(i_{k-1})}f^{(k-1)}$ assumes at most one positive value in $(0,1)$. Since $f^{(k-1)}$ must be continuous on $(0,1)$, $(-1)^{(i_{k-1})}f^{(k-1)} = c \geq 0$ on $(0,1)$.

It should be noted that if f satisfies the hypothesis of Proposition 3.7, then $f^{(n-2)} = 0$ or $f^{(n-2)}$ is extremal in $K(i_{n-2}, i_{n-1}, i_n)$. The results of the last three propositions are summarized in the following theorem which gives a characterization of the extremal elements of $C(i_0, i_1, \ldots, i_n)$ for $n > 2$.

<u>Theorem 3.2</u>: Let $f \in C(i_0, i_1, \ldots, i_n)$, $n > 2$, such that f is not constant and f is continuous on $[0,1]$. Then f is an extremal element of $C(i_0, i_1, \ldots, i_n)$ if, and only if, f is linear on $[0,1]$ and $f(a_0) = 0$; or there is an integer k, $3 \leq k \leq n-1$, such that $f^{(k-2)}$ is extremal in $K(i_{k-2}, i_{k-1}, i_k)$, $f^{(p)}(a_p) = 0$ for $0 \leq p \leq k-3$ and $f^{(k-1)}$ is constant on $(0,1)$; or $f^{(n-2)}$ is an extremal element of $K(i_{n-2}, i_{n-1}, i_n)$ and $f^{(p)}(a_p) = 0$ for $0 \leq p \leq n-3$.

Proof: If f is extremal in $C(i_0, i_1, \ldots, i_n)$, then there is a unique integer k, $2 \leq k \leq n$, such that f is extremal in $C(i_0, \ldots, i_k)$ but is not extremal in $C(i_0, \ldots, i_{k-1})$. If $k \leq n-1$, then f is extremal in $C(i_0, \ldots, i_{n-1})$ and the conclusion follows by Proposition 3.7. If $k = n$, then $f^{(p)}(a_p) = 0$ for $0 \leq p \leq n-3$ and $f^{(n-2)}$ is extremal in $K(i_{n-2}, i_{n-1}, i_n)$ by Lemma 3.5 and Proposition 3.6, respectively.

The proof of the converse will be given in three parts. First, if f is linear on $[0,1]$ and $f(a_0) = 0$, then f is extremal in $C(i_0,i_1,i_2)$ and $\Delta_h^2 f(x) = 0$ for $h > 0$, $[x,x+2h] \subset [0,1]$. It follows that $\Delta_h^k f(x) = 0$ for $h > 0$, $[x,x+kh] \subset [0,1]$, $2 \leq k \leq n$, and thus, f is in $C(i_0,i_1,\ldots,i_n)$. Since $C(i_0,i_1,\ldots,i_n)$ is a subcone of $C(i_0,i_1,i_2)$, f is again extremal in $C(i_0,i_1,\ldots,i_n)$.

In the second place, if $f^{(k-2)}$ is extremal in $K(i_{k-2},i_{k-1},i_k)$, $f^{(p)}(a_p) = 0$ for $0 \leq p \leq k-3$ and $f^{(k-1)}$ is constant on $(0,1)$, where $3 \leq k \leq n-1$, then $f = I(f^{(k-2)},i_0,\ldots,i_{k-2};)$ and f is extremal in $C(i_0,\ldots,i_k)$ by Proposition 3.5. Since $f^{(k-2)}$ is extremal in $K(i_{k-2},i_{k-1},i_k)$, $(-1)^{(i_{k-1})}f^{(k-1)}$ assumes at most one positive value on $(0,1)$ (cf. Proposition 3.4 and Theorem 3.1). It follows that $(-1)^{(i_{k-1})}f^{(k-1)} = c \geq 0$ and thus, $f^{(k-2)}$ is either linear or constant on $(0,1)$. It was shown in the proof of Lemma 3.4 that $\Delta_h^{k-2} f(x) = f^{(k-2)}(\xi)$ for $0 \leq x < \xi < x+(k-2)h \leq 1$, from which it follows that

$$\Delta_h^k f(x) = \Delta_h^2 \Delta_h^{k-2} f(x) = \Delta_h^2 f^{(k-2)}(\xi) = 0$$

for $h > 0$, $[x,x+kh] \subset [0,1]$. Hence, $\Delta_h^p f(x) = 0$ for $h > 0$, $[x,x+ph] \subset [0,1]$, where $k \leq p \leq n$, and thus $f \in C(i_0,i_1,\ldots,i_n)$. Since $C(i_0,i_1,\ldots,i_n)$ is a subcone of $C(i_0,\ldots,i_k)$, f is extremal in $C(i_0,i_1,\ldots,i_n)$.

Thirdly, if $f^{(p)}(a_p) = 0$ for $0 \leq p \leq n-3$, then

$$f = I(f^{(n-2)},i_0,i_1,\ldots,i_{n-2};).$$

Since $f^{(n-2)}$ is extremal in $K(i_{n-2},i_{n-1},i_n)$, f is an extremal element of $C(i_0,i_1,\ldots,i_n)$ by Proposition 3.5.

CHAPTER IV

INTEGRAL REPRESENTATIONS OF n-MONOTONE FUNCTIONS

The set of functions $C(i_0,\ldots,i_n)-C(i_0,\ldots,i_n)$, $n \geq 1$, forms the smallest linear space containing the convex cone $C(i_0,\ldots,i_n)$ (cf. [3], p. 47). If

$$\mathfrak{U} = \{U(\rho,\varepsilon) : \rho \text{ finite subset of } [0,1] \text{ and } \varepsilon > 0\},$$

where

$$U(\rho,\varepsilon) = \{f \in C(i_0,\ldots,i_n)-C(i_0,\ldots,i_n) : |f(x)| < \varepsilon \text{ for } x \in \rho\},$$

then $\mathfrak{U}$ forms a local base at 0 and $C(i_0,\ldots,i_n)-C(i_0,\ldots,i_n)$ together with the local base $\mathfrak{U}$ is a Hausdorff locally convex space. The topology induced by $\mathfrak{U}$ is called the topology of simple convergence and is equivalent to the topology of pointwise convergence (cf. [10], p. 155). It is the purpose of this chapter to prove that the extremal elements of $C(i_0,\ldots,i_n)$ form a closed set in a compact convex set which meets every ray of the cone $C(i_0,\ldots,i_n)$ but does not contain the origin, and to show that for the functions of the cone an integral representation in terms of extremal elements is possible. Since

$$C(1,i_1,\ldots,i_n) = -C(0,1-i_1,\ldots,1-i_n),$$

it is sufficient to obtain these results for the case where $i_0 = 0$.

If

$$S(0,i_1,\ldots,i_n) = \{f \in C(0,i_1,\ldots,i_n) : f(1-a_0) = 1\},$$

where $n \geq 1$, then $S(0,i_1,\ldots,i_n)$ is a convex set which meets every ray of $C(0,i_1,\ldots,i_n)$ once and only once but does not contain the origin, that is the zero function. It then follows that f is an extreme point of $S(0,i_1,\ldots,i_n)$ if, and only if, f is an extremal element of $C(0,i_1,\ldots,i_n)$ which lies in $S(0,i_1,\ldots,i_n)$ (cf. [4], p. 235).

Compactness of $S(0,i_1,\ldots,i_n)$

Since each function f in $S(0,i_1,\ldots,i_n)$ is nonnegative and monotonic, then $0 \leq f(x) \leq 1$ for $x \in [0,1]$. If $I = [0,1]$, then it follows from the Tychonoff theorem that

$$I^I = \{f:[0,1] \to [0,1] : f \text{ is a function}\}$$

with the product topology is a compact space. Since the topology of simple convergence is equivalent to the product topology, it follows that $S(0,i_1,\ldots,i_n)$ can be imbedded in I^I. Therefore, in order to prove that $S(0,i_1,\ldots,i_n)$ is compact, it is sufficient to show that $S(0,i_1,\ldots,i_n)$ is a closed set. This will be done in the following proposition by showing the complement of $S(0,i_1,\ldots,i_n)$ is open.

<u>Proposition 4.1</u>: The set $S(0,i_1,\ldots,i_n)$, $n \geq 1$, is closed.

Proof: If $g \in C(0,i_1,\ldots,i_n)\backslash S(0,i_1,\ldots,i_n)$, then $g(1-a_0) \neq 1$ and the set

$$g + U(\{1-a_0\},\varepsilon)$$
$$= \{f \in C(0,i_1,\ldots,i_n)-C(0,i_1,\ldots,i_n) : |f(1-a_0) - g(1-a_0)| < \varepsilon\},$$

where $\varepsilon = (1/2)|1-g(1-a_0)|$, is an open set about g that fails to meet $S(0,i_1,\ldots,i_n)$. If $g \notin C(0,i_1,\ldots,i_n)$, then there are numbers x_0, k and h such that

$$(-1)^{(i_k)}\Delta_h^k g(x_0) = (-1)^{(i_k)} \sum_{j=0}^{k} (-1)^j \binom{k}{j} g(x_0+(k-j)h) = \delta < 0.$$

Let

$$U = g + U(\{x_0,x_0+h,\ldots,x_0+kh\},\varepsilon)$$
$$= \{f\varepsilon C(0,i_1,\ldots,i_n)-C(0,i_1,\ldots,i_n): |f(x_0+jh)-g(x_0+jh)|<\varepsilon,\ 0 \leq j \leq k\},$$

where $\varepsilon = (1/2)^k(-\delta)$. If $f \in U$, then

$$\begin{aligned}
(-1)^{(i_k)}\Delta_h^k f(x_0) &= (-1)^{(i_k)}\Delta_h^k [f(x_0) - g(x_0)] + (-1)^{(i_k)}\Delta_h^k g(x_0) \\
&\leq |\Delta_h^k [f(x_0) - g(x_0)]| + (-1)^{(i_k)}\Delta_h^k g(x_0) \\
&\leq \sum_{j=0}^{k} \binom{k}{j} |f(x_0+(k-j)h) - g(x_0+(k-j)h)| + \delta \\
&< \varepsilon \sum_{j=0}^{k} \binom{k}{j} + \delta \\
&= 2^k\varepsilon + \delta \\
&= 0,
\end{aligned}$$

and it follows that $f \notin S(0,i_1,\ldots,i_n)$. Hence, if $g \notin S(0,i_1,\ldots,i_n)$, then $g \notin \text{cl}[S(0,i_1,\ldots,i_n)]$ which implies that $S(0,i_1,\ldots,i_n)$ is closed.

Since $C(0,i_1,\ldots,i_n)$-$C(0,i_1,\ldots,i_n)$ is a Hausdorff space and $S(0,i_1,\ldots,i_n)$ is compact, then ext $S(0,i_1,\ldots,i_n)$ is compact if, and only if, ext $S(0,i_1,\ldots,i_n)$ is closed relative to $S(0,i_1,\ldots,i_n)$. The proof that ext $S(0,i_1,\ldots,i_n)$ is closed relative to $S(0,i_1,\ldots,i_n)$ will be by induction on n. If $f \in S(0,i_1)$ such that f is not an extremal element of $C(0,i_1)$, then there exists a number $x_0 \in [0,1]$ such that $0 < f(x_0) < f(1-a_0) = 1$ (cf. Proposition 3.1). Let $U = f + U(\{x_0\},\varepsilon)$, where $\varepsilon = \min\{f(x_0), 1-f(x_0)\}$. If g is an extreme point of $S(0,i_1)$ (that is, g is an extremal element of $C(0,i_1)$ which lies in $S(0,i_1)$), then g assumes exactly one positive value in $[0,1]$, and since $g(1-a_0) = 1$, $g(x)$ is either 0 or 1 for each $x \in [0,1]$. If $g(x_0) = 0$, then $f(x_0) - g(x_0) = f(x_0) \geq \varepsilon$, whereas if $g(x_0) = 1$, then $g(x_0) - f(x_0) = 1 - f(x_0) \geq \varepsilon$. Therefore, $g \notin U$ and it follows that ext $S(0,i_1)$ is closed, where ext $S(0,i_1)$ denotes the set of extreme points of $S(0,i_1)$.

Closure of ext $S(0,i_1,i_2)$

If it has been shown that ext $S(0,i_1,\ldots,i_n)$ is closed, where $n \geq 1$, then

$$\begin{aligned}
&\text{cl}[\text{ext } S(0,i_1,\ldots,i_n,i_{n+1})] \\
&\quad = \text{cl}[\text{ext } S(0,i_1,\ldots,i_{n+1}) \setminus \text{ext } S(0,i_1,\ldots,i_n)] \\
&\qquad\qquad \cup \text{cl}\{[\text{ext } S(0,i_1,\ldots,i_n)] \cap S(0,i_1,\ldots,i_{n+1})\} \\
&\quad = \text{cl}[\text{ext } S(0,i_1,\ldots,i_{n+1}) \setminus \text{ext } S(0,i_1,\ldots,i_n)] \\
&\qquad\qquad \cup \{[\text{ext } S(0,i_1,\ldots,i_n)] \cap S(0,i_1,\ldots,i_{n+1})\},
\end{aligned}$$

since $S(0,i_1,\ldots,i_{n+1})$ is closed. It follows that

$$\begin{aligned} &\mathrm{cl}[\mathrm{ext}\ S(0,i_1,\ldots,i_n,i_{n+1})] \\ &\qquad \subset \mathrm{cl}[\mathrm{ext}\ S(0,i_1,\ldots,i_{n+1}) \setminus \mathrm{ext}\ S(0,i_1,\ldots,i_n)] \\ &\qquad\qquad\qquad\qquad \cup\ \mathrm{ext}\ S(0,i_1,\ldots,i_{n+1}). \end{aligned}$$

Therefore, in order to show ext $S(0,i_1,\ldots,i_{n+1})$ is closed, it is sufficient to prove that

$$\mathrm{cl}[\mathrm{ext}\ S(0,i_1,\ldots,i_{n+1}) \setminus \mathrm{ext}\ S(0,i_1,\ldots,i_n)] \subset \mathrm{ext}\ S(0,i_1,\ldots,i_{n+1}).$$

<u>Proposition 4.2</u>: The set ext $S(0,0,0)$ is closed.

Proof: In view of the above remarks, it suffices to show that cl[ext $S(0,0,0) \setminus$ ext $S(0,0)] \subset$ ext $S(0,0,0)$. If

$$f \in \mathrm{cl}[\mathrm{ext}\ S(0,0,0) \setminus \mathrm{ext}\ S(0,0)],$$

then $f \in S(0,0,0)$ by Proposition 4.1. There is a sequence $\{f_i\}$ of functions converging pointwise to f on $[0,1]$ such that each f_i is an extreme point of $S(0,0,0)$ which is not extreme in $S(0,0)$. It follows from Theorem 3.1 that $f_i = m_i f(\xi_i,0,0;)$ such that $f_i(1) = 1$; that is, $f_i(x) = 0$, $x \in [0,\xi_i]$ and $(1-\xi_i)^{-1}(x-\xi_i)$ for $x \in [\xi_i,1]$, where $0 \leq \xi_i < 1$. If the sequence $\{\xi_i\}$ of real numbers converges to 1, then it is easily seen that $\lim\limits_{i\to\infty} f_i(x) = 0$, for $x \in [0,1)$ while $\lim\limits_{i\to\infty} f_i(1) = 1$. Since the topology on $C(0,0,0)$-$C(0,0,0)$ is Hausdorff, the sequence $\{f_i\}$ of functions has a unique pointwise limit. It follows that $f(x) = 0$, for $x \in [0,1)$ and $f(1) = 1$ and f is therefore an extreme point of $S(0,0)$. Since $f \in S(0,0,0)$, f is again extreme in $S(0,0,0)$.

On the other hand, if $\{\xi_i\}$ does not converge to 1, then there is a nonnegative number $\xi_0 < 1$ and a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to ξ_0. If $0 \leq x < \xi_0$, then

$$\lim_{j \to \infty} f_j(x) = 0;$$

whereas

$$\lim_{j \to \infty} f_j(x) = \frac{1}{1-\xi_0}(x-\xi_0),$$

if $\xi_0 \leq x \leq 1$. Therefore, since the topology is Hausdorff, $f(x) = 0$, $x \in [0,\xi_0]$ and $(1-\xi_0)^{-1}(x-\xi_0)$ for $x \in [\xi_0,1]$. Hence, $f = (1-\xi_0)^{-1}f(\xi_0,0,0;)$ and it follows from Theorem 3.1 that f is in ext $S(0,0,0)$.

<u>Corollary 4.1</u>: The set ext $S(0,1,0)$ is closed.

Proof: The corollary follows from Proposition 4.2 by noting that $f \in S(0,1,0)$ if, and only if, $g \in S(0,0,0)$, where $g(x) = f(1-x)$ for $x \in [0,1]$.

<u>Corollary 4.2</u>: The set ext $S(0,1,1)$ is closed.

Proof: If $f \in \mathrm{cl}[\mathrm{ext}\, S(0,1,1) \setminus \mathrm{ext}\, S(0,1)]$, then $f \in S(0,1,1)$ by Proposition 4.1 and there is a sequence $\{f_i\}$ of functions which are extreme in $S(0,1,1)$ but not extreme in $S(0,1)$ converging to f pointwise on $[0,1]$. It follows from Theorem 3.1 that $f_i = m_i f(\xi_i,1,1;)$ such that $f_i(0) = 1$; that is, $f_i(x) = 1$, $x \in [0,\xi_i]$ and $(1-\xi_i)^{-1}(1-x)$ for $x \in [\xi_i,1]$, where $0 \leq \xi_i < 1$. Since $1-f_i = (1-\xi_i)^{-1}f(\xi_i,0,0;)$ (that is, $1-f_i(x) = 0$, $x \in [0,\xi_i]$ and $(1-\xi_i)^{-1}(x-\xi_i)$ for $x \in [\xi_i,1]$), then $1-f_i \in \mathrm{ext}\, S(0,0,0) \setminus \mathrm{ext}\, S(0,0)$, for each i. It follows from

Proposition 4.2 that $1-f \varepsilon$ ext $S(0,0,0)$, since $1-f$ is the pointwise limit of the sequence $\{1-f_i\}$ on $[0,1]$, which implies that f is in ext $S(0,1,1)$. Hence, by the remarks preceding Proposition 4.2, ext $S(0,1,1)$ is closed.

Corollary 4.3: The set ext $S(0,0,1)$ is closed.

Proof: The corollary follows easily from Corollary 4.2 by noting that $f \varepsilon S(0,0,1)$ if, and only if, $g \varepsilon S(0,1,1)$, where $g(x) = f(1-x)$ for $x \varepsilon [0,1]$.

It follows from Proposition 4.2 and Corollaries 4.1, 4.2 and 4.3 that ext $S(0,i_1,i_2)$ is a closed set, where $i_j = 0$ or 1, $j = 1,2$.

Closure of ext $S(0,i_1,\ldots,i_n)$, $n > 2$

It has already been noted that in order to show ext $S(0,i_1,\ldots,i_n)$ is closed, where $n > 2$, it is sufficient to prove

$$\mathrm{cl}[\text{ext } S(0,i_1,\ldots,i_n) \setminus \text{ext } S(0,i_1,\ldots,i_{n-1})] \subset \text{ext } S(0,i_1,\ldots,i_n).$$

If $f \varepsilon \mathrm{cl}[\text{ext } S(0,i_1,\ldots,i_n) \setminus \text{ext } S(0,i_1,\ldots,i_{n-1})]$, then f is in $S(0,i_1,\ldots,i_n)$ by Proposition 4.1 and there is a sequence $\{f_i\}$ of functions in ext $S(0,i_1,\ldots,i_n) \setminus$ ext $S(0,i_1,\ldots,i_{n-1})$ which converges pointwise to f on $[0,1]$. It follows from Proposition 3.6 that $f_i^{(n-2)}$ is a nonconstant extremal element of $K(i_{n-2},i_{n-1},i_n)$. By using Proposition 3.4, Lemma 3.5 and Theorem 3.1, this implies that $f_i^{(n-2)} = m_i f(\xi_i,i_{n-1},i_n;)$ for $i_{n-2} = 0$, $f_i^{(n-2)} = -m_i f(\xi_i,1-i_{n-1},1-i_n;)$ for $i_{n-2} = 1$ and $f_i = I(f_i^{(n-2)},0,i_1,\ldots,i_{n-2};)$, where $m_i > 0$ and $\xi_i \varepsilon (0,1)$ or $\xi_i = a_{n-1} = (1/2)[1-(-1)^{(i_{n-1}+i_n)}]$, for each i, since

$K(1,i_{n-1},i_n) = -K(0,1-i_{n-1},1-i_n)$. These observations are summarized in the following remark.

Remark 4.1: In order to show ext $S(0,i_1,\ldots,i_n)$, $n > 2$, is closed, it suffices to prove that $f \in$ ext $S(0,i_1,\ldots,i_n)$ whenever f is the pointwise limit on $[0,1]$ of a sequence $\{f_i\}$ of functions with the property that

$$f_i^{(n-2)} = m_i f(\xi_i,i_{n-1},i_n;)$$

for $i_{n-2} = 0$,

$$f_i^{(n-2)} = -m_i f(\xi_i,1-i_{n-1},1-i_n;)$$

for $i_{n-2} = 1$, $f_i(1-a_0) = 1$ and

$$f_i = I(f_i^{(n-2)},0,i_1,\ldots,i_{n-2};),$$

where $\xi_i \in (0,1)$ or $\xi_i = a_{n-1} = (1/2)[1-(-1)^{(i_{n-1}+i_n)}]$ and $m_i > 0$, for each i. The function $mf(\xi,i_1,i_2;)$ is given by equation 3.1 on page 33.

It will be shown, by considering every possible case, that ext $S(0,i_1,i_2,i_3)$ is closed. Then it will be shown that the same technique still can be used to prove ext $S(0,i_1,\ldots,i_n)$ is closed for $n > 3$.

Proposition 4.3: The set ext $S(0,0,0,0)$ is closed.

Proof: Let $\{f_i\}$ be a sequence of functions in $S(0,0,0,0)$ converging pointwise to a function f on $[0,1]$ such that $f_i = I(f_i',0,0;)$ and $f_i' = m_i f(\xi_i,0,0;)$; that is, $f_i'(x) = 0$, $x \in [0,\xi_i]$ and $m_i(x-\xi_i)$ for $x \in [\xi_i,1]$, where $m_i > 0$ and $0 \leq \xi_i < 1$. Since $f_i = I(f_i',0,0;)$

and $f_i(1) = 1$, it follows that $f_i(x) = 0$, $x \in [0,\xi_i]$ and

$$f_i(x) = \frac{1}{(1-\xi_i)^2}(x-\xi_i)^2$$

for $x \in [\xi_i,1]$. If the sequence $\{\xi_i\}$ of real numbers converges to 1, then it is easily seen that

$$\operatorname*{limit}_{i\to\infty} f_i(x) = 0$$

for $x \in [0,1)$ and

$$\operatorname*{limit}_{i\to\infty} f_i(1) = 1.$$

Since the topology of simple convergence is Hausdorff, the sequence $\{f_i\}$ has a unique pointwise limit and it follows that $f(x) = 0$, $x \in [0,1)$ and $f(1) = 1$. Since f is an extreme point of S(0,0,0) which is in S(0,0,0,0), f is again extreme in S(0,0,0,0).

If the sequence $\{\xi_i\}$ does not converge to 1, then there is a number $\xi_0 \in [0,1)$ and a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to ξ_0. It will be shown that the subsequence $\{f_j\}$ of $\{f_i\}$ converges uniformly on $[0,1]$ to an extreme point of S(0,0,0,0). Then, since the topology is Hausdorff, it must follow that $f \in \text{ext } S(0,0,0,0)$.

If ε is a positive number and $\delta = (1/2)(1-\xi_0)^2\varepsilon$, then there is a positive integer N such that $|\xi_j-\xi_0| < \delta$ whenever $j > N$. Define the function g such that $g(x) = 0$, $x \in [0,\xi_0]$ and

$$g(x) = \frac{2}{(1-\xi_0)^2}(x-\xi_0)$$

for $x \in [\xi_0, 1]$. If $\xi_j = \xi_0$, for some j, then $f'_j = g$. If $j > N$, $\xi_j < \xi_0$ and $x \in [0, \xi_j]$, then $|f'_j(x) - g(x)| = 0$. If $j > N$, $\xi_j < \xi_0$ and $x \in (\xi_j, \xi_0)$, then

$$|f'_j(x) - g(x)| = 2\frac{x-\xi_j}{(1-\xi_j)^2} \leq 2\frac{\xi_0-\xi_j}{(1-\xi_j)^2} < 2\frac{\xi_0-\xi_j}{(1-\xi_0)^2} < \varepsilon.$$

If $j > N$, $\xi_j < \xi_0$ and $x \in [\xi_0, 1]$, then

$$|f'_j(x) - g(x)| = \left| 2\frac{x-\xi_0}{(1-\xi_0)^2} - 2\frac{x-\xi_j}{(1-\xi_j)^2} \right|;$$

but since f'_j and g are linear on $[\xi_0, 1]$, then

$$\begin{aligned} |f'_j(x) - g(x)| &\leq \max\{|f'_j(\xi_0) - g(\xi_0)|, |f'_j(1) - g(1)|\} \\ &\leq \max\{2\frac{\xi_0-\xi_j}{(1-\xi_j)^2}, \frac{2}{1+\xi_j} - \frac{2}{1+\xi_0}\} \\ &\leq \max\{2\frac{\xi_0-\xi_j}{(1-\xi_0)^2}, 2(\xi_0-\xi_j)\} \\ &= 2\frac{\xi_0-\xi_j}{(1-\xi_0)^2} \\ &< \varepsilon. \end{aligned}$$

By a completely analogous argument, it can be shown that $|f'_j(x)-g(x)|<\varepsilon$ for $x \in [0,1]$, $j > N$ and $\xi_j > \xi_0$. Thus, the sequence $\{f'_j\}$ converges uniformly to the function g on [0,1]. Therefore, the sequence $\{f_j\}$ converges uniformly to the function I(g,0,0;) on [0,1] because $f_j = I(f'_j,0,0;)$. Since g is an extremal element of K(0,0,0) and I(g,0,0;1) = 1, it follows from Proposition 3.5 that I(g,0,0;) is an extreme point of $S(0,0,0,0)$. Hence, $f \varepsilon$ ext $S(0,0,0,0)$ since $f = I(g,0,0;)$, and it follows from Remark 4.1 that ext $S(0,0,0,0)$ is closed.

Corollary 4.4: The set ext $S(0,1,0,1)$ is closed.

Proof: The corollary follows easily from Proposition 4.3 by noting that $f \varepsilon S(0,1,0,1)$ if, and only if, $g \varepsilon S(0,0,0,0)$, where $g(x) = f(1-x)$, $x \varepsilon [0,1]$.

Corollary 4.5: The set ext $S(0,1,1,1)$ is closed.

Proof: Let f be the pointwise limit on $[0,1]$ of a sequence $\{f_i\}$ of functions such that $f_i = I(f_i',0,1;)$ and $f_i(0) = 1$ for each i, where $f_i' = -m_i f(\xi_i,0,0;)$ for $m_i > 0$ and $\xi_i \varepsilon [0,1)$. Then $f_i(x) = 1$, $x \varepsilon [0,\xi_i]$ and $1-(1-\xi_i)^{-2}(x-\xi_i)^2$ for $x \varepsilon [\xi_i,1]$, and by Remark 4.1, it suffices to show $f \varepsilon$ ext $S(0,1,1,1)$. Since $1-f_i(x) = 0$, $x \varepsilon [0,\xi_i]$ and $(1-\xi_i)^{-2}(x-\xi_i)^2$ for $x \varepsilon [\xi_i,1]$, then

$$1-f_i \ \varepsilon \text{ ext } S(0,0,0,0) \setminus \text{ext } S(0,0,0)$$

for each i. Since $1-f$ is the pointwise limit of the sequence $\{1-f_i\}$ on $[0,1]$, it follows from Proposition 4.3 that $1-f \varepsilon$ ext $S(0,0,0,0)$. Hence, $f \varepsilon$ ext $S(0,1,1,1)$ and ext $S(0,1,1,1)$ is closed.

Corollary 4.6: The set ext $S(0,0,1,0)$ is closed.

Proof: Since $f \varepsilon S(0,0,1,0)$ if, and only if, $g \varepsilon S(0,1,1,1)$, where $g(x) = f(1-x)$ for $x \varepsilon [0,1]$, it follows from Corollary 4.5 that ext $S(0,0,1,0)$ is closed.

The remaining cases for $n = 3$ follow from the next proposition.

<u>Proposition 4.4</u>: The set ext $S(0,0,0,1)$ is closed.

Proof: Let $\{f_i\}$ be a sequence of functions in $S(0,0,0,1)$ converging pointwise to a function f on $[0,1]$ such that $f_i = I(f_i',0,0;)$ and $f_i' = m_i f(\xi_i,0,1;)$; that is, $f_i(x) = m_i x$, $x \varepsilon [0,\xi_i]$ and $m_i \xi_i$ for $x \varepsilon [\xi_i,1]$, where $m_i > 0$ and $0 < \xi_i \leq 1$. Since $f_i = I(f_i',0,0;)$ and $f_i(1) = 1$ for each i, it follows that

$$f_i(x) = \frac{1}{\xi_i(2-\xi_i)} x^2,$$

$x \varepsilon [0,\xi_i]$ and

$$f_i(x) = \frac{1}{2-\xi_i} (2x-\xi_i)$$

for $x \varepsilon [\xi_i,1]$. If the sequence $\{\xi_i\}$ of real numbers converges to 0, then it is easily seen that

$$\operatorname*{limit}_{i \to \infty} f_i(x) = x$$

for $x \varepsilon [0,1]$. Since the topology of simple convergence is Hausdorff, the pointwise limit of the sequence $\{f_i\}$ is unique and it follows that $f(x) = x$, $x \varepsilon [0,1]$. Hence, f is an extreme point of $S(0,0,0)$ which implies that f is again extreme in $S(0,0,0,1)$.

If the sequence $\{\xi_i\}$ does not converge to 0, then there is a number $\xi_0 \varepsilon (0,1]$ and a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to ξ_0. It will be shown that the subsequence $\{f_j\}$ of $\{f_i\}$ converges uniformly on $[0,1]$ to an extreme point of $S(0,0,0,1)$. Then, since the topology is Hausdorff, it must follow that $f \varepsilon$ ext $S(0,0,0,1)$.

If ε is a positive number and $\delta = (1/4)\xi_0\varepsilon$, then there is a positive integer N such that $|\xi_j - \xi_0| < \delta$ whenever $j > N$. Define the function g such that

$$g(x) = \frac{2}{\xi_0(2-\xi_0)}\, x,$$

$x \in [0,\xi_0]$ and

$$g(x) = \frac{2}{2-\xi_0}$$

for $x \in [\xi_0, 1]$. If $\xi_j = \xi_0$, for some j, then $f'_j = g$. If $j > N$, $\xi_j > \xi_0$ and $x \in [0,\xi_0]$, then

$$|f'_j(x) - g(x)| = \left|\frac{2}{\xi_j(2-\xi_j)}\, x - \frac{2}{\xi_0(2-\xi_0)}\, x\right|;$$

but since $f'_j(0) = g(0) = 0$ and f'_j and g are linear on $[0,\xi_0]$, then

$$\begin{aligned}
|f'_j(x) - g(x)| &\leq |f'_j(\xi_0) - g(\xi_0)| \\
&= \left|\frac{2\xi_0}{\xi_j(2-\xi_j)} - \frac{2}{2-\xi_0}\right| \\
&= \frac{2}{\xi_j}\left|\frac{\xi_0}{2-\xi_j} - \frac{\xi_j}{2-\xi_0}\right| \\
&\leq \frac{2}{\xi_j}\, |2(\xi_0-\xi_j) - (\xi_0^2-\xi_j^2)| \\
&= \frac{2}{\xi_j}\, (\xi_j-\xi_0)\, [2-(\xi_0+\xi_j)] \\
&< \frac{4}{\xi_j}\, (\xi_j-\xi_0) \\
&< \frac{4}{\xi_0}\, (\xi_j-\xi_0) \\
&< \varepsilon. \qquad (4.1)
\end{aligned}$$

If $j > N$, $\xi_j > \xi_0$ and $x \in [\xi_j,1]$, then

$$|f_j'(x) - g(x)| = \left|\frac{2}{2-\xi_j} - \frac{2}{2-\xi_0}\right| \leq 2(\xi_j-\xi_0) < \varepsilon. \tag{4.2}$$

If $j > N$, $\xi_j > \xi_0$ and $x \in (\xi_0,\xi_j)$, then

$$|f_j'(x) - g(x)| \leq \max\{|f_j'(\xi_0) - g(\xi_0)|, |f_j'(\xi_j) - g(\xi_j)|\}$$

$$= \max\left\{\left|\frac{2\xi_0}{\xi_j(2-\xi_j)} - \frac{2}{2-\xi_0}\right|, \left|\frac{2}{2-\xi_j} - \frac{2}{2-\xi_0}\right|\right\},$$

since f_j' is linear and g is constant on (ξ_0,ξ_j), and it follows from inequalities 4.1 and 4.2 that $|f_j'(x) - g(x)| < \varepsilon$. By a completely analogous argument, it can be shown that $|f_j'(x) - g(x)| < \varepsilon$ for $x \in [0,1]$, $j > N$ and $\xi_j < \xi_0$. Thus, the sequence $\{f_j'\}$ converges uniformly to the function g on $[0,1]$. Therefore, the sequence $\{f_j\}$ converges uniformly to $I(g,0,0;)$ on $[0,1]$ because $f_j = I(f_j',0,0;)$. Since g is an extremal element of $K(0,0,1)$ and $I(g,0,0;1) = 1$, it follows from Proposition 3.5 that $I(g,0,0;)$ is an extreme point of $S(0,0,0,1)$. Hence, $f \in \text{ext}\, S(0,0,0,1)$ and it follows from Remark 4.1 that $\text{ext}\, S(0,0,0,1)$ is closed.

Corollary 4.7: The set $\text{ext}\, S(0,1,0,0)$ is closed.

Proof: The corollary follows easily from Proposition 4.4 by noting that $f \in S(0,1,0,0)$ if, and only if, $g \in S(0,0,0,1)$, where $g(x) = f(1-x)$ for $x \in [0,1]$.

Corollary 4.8: The set $\text{ext}\, S(0,1,1,0)$ is closed.

Proof: Let f be the pointwise limit on $[0,1]$ of a sequence $\{f_i\}$ of functions such that $f_i = I(f_i',0,1;)$ and $f_i(0) = 1$ for each i, where $f_i' = -m_i f(\xi_i,0,1;)$ for $m_i > 0$ and $\xi_i \in (0,1]$. Then

$$f_i(x) = 1 - \frac{1}{\xi_i(2-\xi_i)} x^2,$$

$x \in [0,\xi_i]$ and

$$f_i(x) = \frac{2}{2-\xi_i}(1-x)$$

for $x \in [\xi_i,1]$, and by Remark 4.1, it suffices to show that f is in ext $S(0,1,1,0)$. Since $1-f_i(x) = [\xi_i(2-\xi_i)]^{-1}x^2$, $x \in [0,\xi_i]$ and $(2-\xi_i)^{-1}(2x-\xi_i)$ for $x \in [\xi_i,1]$, then

$$1-f_i \in \text{ext } S(0,0,0,1) \setminus \text{ext } S(0,0,0)$$

for each i. Since 1-f is the pointwise limit of the sequence $\{1-f_i\}$ on $[0,1]$, it follows from Proposition 4.4 that $1-f \in$ ext $S(0,0,0,1)$. Hence, $f \in$ ext $S(0,1,1,0)$ and ext $S(0,1,1,0)$ is closed.

<u>Corollary 4.9</u>: The set ext $S(0,0,1,1)$ is closed.

Proof: Since $f \in S(0,0,1,1)$ if, and only if, $g \in S(0,1,1,0)$, where $g(x) = f(1-x)$ for $x \in [0,1]$, it follows from Corollary 4.8 that ext $S(0,0,1,1)$ is closed.

It follows from Propositions 4.3 and 4.4 and Corollaries 4.4 - 4.9 that ext $S(0,i_1,i_2,i_3)$ is closed. The proof that ext $S(0,i_1,\ldots,i_n)$, $n > 3$, is closed is essentially the same as that for $n = 3$. Let f be the pointwise limit on $[0,1]$ of a sequence $\{f_i\}$ of functions with the property that

$$f_i^{(n-2)} = m_i f(\xi_i, i_{n-1}, i_n;)$$

for $i_{n-2} = 0$,

$$f_i^{(n-2)} = -m_i f(\xi_i, 1-i_{n-1}, 1-i_n;)$$

for $i_{n-2} = 1$, $f_i(1-a_0) = 1$ and

$$f_i = I(f_i^{(n-2)}, 0, i_1, \ldots, i_{n-2};),$$

where $m_i > 0$ and $\xi_i \in (0,1)$ or $\xi_i = a_{n-1} = (1/2)[1-(-1)^{(i_{n-1}+i_n)}]$, for each i and $n > 3$. If the sequence $\{\xi_i\}$ of real numbers converges to $1-a_{n-1}$, then as in the first part of the proofs of both Proposition 4.3 and Proposition 4.4, it can be shown that $f \in \text{ext } S(0, i_1, \ldots, i_{n-1})$ and f is again extreme in $S(0, i_1, \ldots, i_n)$.

On the other hand, if the sequence $\{\xi_i\}$ does not converge to $1-a_{n-1}$, then there is a real number $\xi_0 \in [0,1] \setminus \{1-a_{n-1}\}$ and a subsequence $\{\xi_j\}$ of $\{\xi_i\}$ such that $\{\xi_j\}$ converges to ξ_0. If $i_{n-2} = i_n$ ($i_{n-2} \neq i_n$), then the technique used in the latter part of the proof of Proposition 4.3 (Proposition 4.4) can again be used to show that the subsequence $\{f_j^{(n-2)}\}$ of $\{f_i^{(n-2)}\}$ converges uniformly to an extremal element g of $K(i_{n-2}, i_{n-1}, i_n)$. In either case, it must follow that $\{f_j\}$ converges uniformly to $I(g, 0, i_1, \ldots, i_{n-2};)$ on $[0,1]$ because $f_j = I(f_j^{(n-2)}, 0, i_1, \ldots, i_{n-2};)$. Since the topology of simple convergence is a Hausdorff topology, then $f = I(g, 0, i_1, \ldots, i_{n-2};)$ and it follows from Proposition 3.5 that f is an extreme point of $S(0, i_1, \ldots, i_n)$.

Integral Representations

Since $\text{ext } S(0, i_1, \ldots, i_n)$ and $S(0, i_1, \ldots, i_n)$ are both compact subsets of the locally convex space $C(0,i_1,\ldots,i_n)-C(0,i_1,\ldots,i_n)$, $n \geq 1$, it follows from Theorem 39.4 of Choquet [4] that for any function $f_0 \varepsilon S(0,i_1,\ldots,i_n)$ there exists a probability measure μ_0 on the extreme points of $S(0,i_1,\ldots,i_n)$ such that

$$f_0(x) = \int f(x)\, d\mu_0,$$

for $x \varepsilon [0,1]$. Since $S(0,i_1,\ldots,i_n)$ meets every ray of $C(0,i_1,\ldots,i_n)$ and does not contain the origin, it follows that each function of $C(0,i_1,\ldots,i_n)$ is a scalar multiple of such a representation.

If the extreme points of $S(0,i_1,\ldots,i_n)$ were dense in $S(0,i_1,\ldots,i_n)$; that is, if

$$\text{ext } S(0,i_1,\ldots,i_n) = S(0,i_1,\ldots,i_n),$$

then the integral representation above would be of little value. To see that this is not the case whenever $n \geq 2$, let

$$g(x) = (1/2) + (1/2)f(x),$$

$x \varepsilon [0,1]$, where

$$f \varepsilon \text{ ext } S(0,i_1,\ldots,i_n) \setminus \text{ext } S(0,i_1,\ldots,i_{n-1}).$$

Then $g \varepsilon S(0,i_1,\ldots,i_n) \setminus \text{ext } S(0,i_1,\ldots,i_n)$ since g is not constant and $g(a_0) = (1/2) > 0$. Likewise, if

$$g(x) = (1/2) + (1/2)\{(1/2) - (-1)^{(i_1)}[(1/2)-f(x)]\},$$

$x \varepsilon [0,1]$, where $f(x) = 0$, $x \varepsilon [0,(1/2))$ and 1 for $x \varepsilon [(1/2),1]$, then it follows from Proposition 3.1 that $g \varepsilon S(0,i_1) \setminus \text{ext } S(0,i_1)$.

Therefore, the set $S(0,i_1,\ldots,i_n) \setminus \text{ext}\, S(0,i_1,\ldots,i_n)$ is nonempty for $n \geq 1$.

CHAPTER V

SUMMARY

The basic purpose of this study has been to determine the extremal structure of the convex cone of n-monotone functions and to determine the relationships that exist between the extremal elements and the elements of this cone. The extremal elements of the cone of n-monotone functions were completely characterized and it was shown that for any n-monotone function an integral representation in terms of extremal elements is possible.

By using the results of Chapter IV and the Krein-Milman Theorem, it is evident that any n-monotone function can be approximated at a finite number of points in [0,1] by a convex combination of extremal elements of the cone of n-monotone functions. This fact may be useful in the numerical solution of certain difference equations. Another problem of interest is that of characterizing the functions in the linear space $C(i_0,\ldots,i_n)-C(i_0,\ldots,i_n)$, where $n \geq 1$.

There are several problems analogous to the one in this study for which the same type of investigation would be of interest. One such problem is that of considering real-valued functions on a partially ordered semi-group with a smallest element whose first n differences satisfy certain inequalities; for example, the functions might be defined on the half-line $[0,\infty)$. The domain of the functions in question could also be the unit rectangle in E_2 (that is $[0,1] \times [0,1]$).

The n-monotone functions are obtained by specifying the first n differences. A problem of interest would be that of considering the convex cone of real functions on [0,1] where only some of the first n differences are specified. For example, each difference of even order less than n might be specified.

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